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ABSTRACT

This publication was developed as a portion of a two-semester sequence commencing at either the sixth or seventh term of the undergraduate program in electrical engineering at the University of Pittsburgh. The materials of the two courses, produced by a National Science Foundation grant, are concerned with power conversion systems comprising power electronic devices, electromechanical energy converters, and associated logic configurations necessary to cause the system to behave in a prescribed fashion. The emphasis in this portion of the two course sequence (Part 1) is on electric machinery analysis. Techniques applicable to electric machines under dynamic conditions are analyzed. This publication consists of seven chapters which deal with: (1) basic principles; (2) elementary concept of torque and generated voltage; (3) the generalized machine; (4) direct current machines; (5) cross field machines; (6) synchronous machines; and (7) polyphase asynchronous machines. (HM)

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POWER PROCESSING

Part I

"Electric Machinery Analysis"

Howard B. Hamilton
University of Pittsburgh
1970

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GENERAL DESCRIPTION OF PROJECT

The overall objective of the project was development of a model two semester sequence of lecture-laboratory courses, senior year level, to prepare the undergraduate student with the necessary background in electric machinery and to present the fundamental concepts of modeling power processing devices and circuits necessary for future professional activity in Power Processing.

It is felt that the development of this sequence of courses will be of vital interest to educators. It will strengthen the Electrical Engineering curriculum because it will accomplish the following:

1. make the electric machinery and power topics much more interesting and meaningful to the student.
2. better preparation, for the student, for a productive professional career in this vital area.
3. reinforce the learning gleaned from earlier courses in logic theory and solid state electronics.

Courses involving the theory of electric machinery and associated phenomenon have always been an integral part of the electrical engineering undergraduate curriculum. Prior to 1950, colleges and universities were in more or less general agreement concerning the portion of the total curriculum devoted to electric machinery (and power) and to the specific course content. The technological developments associated with the 1940's placed tremendous pressures upon the curriculum. The phenomena associated with new technologies - such as solid state theory, control system analysis etc. had to be included and at the same time, the total time available for the first degree was even decreased in many instances. These pressures resulted in elimination or suppression of the "how-to-do" type courses and also resulted in a reduction of time allotted to certain areas within the curricula. The courses in electric machinery and power were victims of the latter economy.

Usually, the electrical machinery courses (pre 1950) consisted of detailed study of the physical aspects of the particular machine and from these aspects, the input-output characteristics were deduced for the static, or steady state situation. Coincident with the widespread introduction of control system, or servo mechanisms, the time allotted for machinery was again reduced so that the majority of colleges and universities attempted to optimize the machinery offering by presenting the subject matter as if it consisted of devices with input-output terminals and considering that only these aspects of the machine were necessary for study of the machine in dynamic systems. The rationale for this approach was that only a very small percentage of the students were interested in, or would be professionally involved in, the design of the machine. The power courses usually were completely eliminated from the curriculum.

The past few years have seen tremendous strides in the areas of solid state electronics and digital logic theory. The potential applications and useage of these technologies, taken in combination, are just beginning to be appreciated. The useage of these phenomena in electric machinery control and in power systems has opened a new era of technology, i.e., "Power Processing" or "Power Conditioning". As evidence of its importance - and the fact that the whole concept needs research and definition - the National Aeronautics and Space Administration, in its NASA Research Topics Bulletin SC/RTB-12, dated September 14, 1966, stated:

> "Development and application trends in electric power processing (conditioning) indicate a need for further fundamental research to explore and define the principles, and theoretical and physical limitations of related circuits and components. Expected beneficial results of such research would include: a. The formulation of theoretical criteria for physically realizable power processing circuits and systems. b. The provision of meaningful guides and objectives to researchers in related fields (e.g. magnetics, dielectrics, semi-conductors, etc.) and c. The establishment of long-range goals for component development through the rigorous analysis and quantitative identification of performance limiting characteristics of components and related circuits".

Several of the major electrical manufacturers have substantial research efforts underway in this general area. Many of the engineers working in this area are European. Whether it is inclination, desire, availability, or prior education that results in this manpower situation is pure conjecture. The facts that do emerge are that engineering students graduating from U.S. colleges and universities have not had an interest in electric machinery or power, they have not had formal training in the marriage of solid state devices, logic, and power systems, and they are not prepared to move into this important, challenging, aspect of electrical engineering.

As further evidence of professional concern for including this type of material in the education and training of engineers, it should be noted that this is a topic of concern to a group entitled, "Interagency Advanced Power Group". This group has a Power Conditioning Panel (under the Electrical

Working Group). It is composed of representatives from NASA, AFAPl, USAECOM and others. Indicative of their concern for this topic is the "Guest List" at their meeting of 8 December 1966. This list includes representatives from Westinghouse, Honeywell, Boeing, General Electric, TRW Systems, etc. as well as personnel from MIT and Duke University. In the meeting minutes, under paragraphs 2.14 and 2.15 appear the following:

> ... "The ultimate need (for work in this area) is a power systems engineer with broad education in electrical power engineering, including a working knowledge of communications and system theory, who understands the specified requirements and one who can communicate with his colleagues. Students are wary of going into the field of conventional utility type power engineering...they are looking toward more challenging fields, as electric power processing could be if it were properly interpreted and presented.

Thus, it appears that a need does exist for educational courses (not "how-to" type training) that will better prepare undergraduates for productive careers in this field. Few, if any U.S. universities offer work with emphasis in Power Processing but there is discussion and interest, which manifests itself at professional meetings, in the topic.

An awareness of the above stimulated our interest in the course development.

The most descriptive assessment of the final orientation of the project is contained in the prefaces to Part I and Part II. Portions of these prefaces are as follows.

A Quote from Preface to Power Processing, Part I

This material was developed as a portion of a two course sequence concerned with power conversion systems comprising power electronic devices, electro-mechanical energy converters and associated logic configurations necessary to cause the system to behave in a prescribed fashion. The emphasis in this portion of the two course sequence will be on electric machinery analysis.

Electric machines are members of a large class of devices known generally as "electro-mechanical converters". The name imples conversion of energy from electrical to mechanical form - or vice versa. The devices range from very small signal devices, such as microphones, through very large generators supplying as much as 1000 megawatts of power. In some devices, the mechanical motion is linear; in others it is rotary. All devices are based, to an extent, on a common phenomenon and there are many fine textbooks which present a completely generalized approach to electro-mechanical energy conversion. Some of these texts are listed in Appendix I. Time limitations preclude complete coverage of all aspects of electro-mechanical energy conversion. For this reason, the emphasis in this course will be restricted to the analysis techniques applicable to electric machines under dynamic conditions.

There are many facets to the general area of electric machines. At one extreme is the specialized area of design of specific machines. At the other extreme is the application aspect of choosing a machine for a specific use or application. This course will avoid detailed design aspects and will concentrate on analysis of performance. In order to become proficient in analysis, it will be necessary to closely examine some of the internal processes involved in accomplishing the change in energy form, i.e., the torque and voltage producing processes, commutation, leakage, flux, etc., as well as the physical arrangement of various types of machines.

The majority of electrical engineers who come into contact with electric machines are either designers of or analyzers of, systems of which the electric machine is an integral part. Some systems are of the extremely simple "open" type. An example is a simple motor driving a load such as a tool, a pump, or an electric fan. The motor is energized, comes up to speed and performs in the steady state. The control for this system is usually an "on-off" type and the transient performance of the motor is of limited interest. Very little skill is necessary to match morot voltage, speed, and torque ratings to the load requirement. However, as applications become more demanding and the machine becomes part of a complex system, or a process, a knowledge of the transient, or dynamic, behavior is necessary.

The past few years have seen the advent of sophisticated control systems, widespread usage of computers and logic machines, and the development of semiconductor devices with control features and power ratings which have had a tremendous impact on the design of systems involving electric machines. Indeed, a whole new area of specialization has emerged within the electrical engineering field. This is the area of "Power Processing" or "Power Conditioning". Power Processing involves conversion of power or energy from the form and level existing in the available power to the form and level necessary for its utilization in some 'end' device. The ability to make the necessary changes has been considerably enhanced by the development and commercial availability of solid state (semi-conductor) devices. To make use of these devices, certain sequences and combinations of events must be provided for. Thus, logic circuitry and computing machines and their relationship to the semi-conductor devices must also be studied if skill and competence in system design are to be achieved. The other course in this two course sequence concerned itself with this relationship as well as the considerations involved when the logic sub-system - semi-conductor sub-system is utilized to modulate electro-mechanical (electric machine) converters and electrical level converters.

These courses were developed as a two semester sequence commencing at either the sixth or seventh term of the undergraduate program in Electrical Engineering at the University of Pittsburgh.

For the electric machinery portion, student familiarity with the following topics is presumed:

1. electric and magnetic circuits, field theory and transformers.
2. a limited knowledge of linear system analysis and techniques to include Laplace Transform usage function and block diagram representation.
3. solving engineering problems on the analog and digital computer.

Concurrent with the electric machinery portion of the two course sequence the student should have a first course in semi-conductor electronics and an exposure to logic circuitry principles.

Quote from Preface to Power Processing, Part II

This book is intended for use as a text in the senior elective course "Power Processing II", Electrical Engineering Department, University of Pittsburgh. The material presented here has been successfully used as the course content for two trimesters. Although several available books were considered and tried as texts, none were found to be suitable in the light of the objectives of the course, and therefore this book has been written to fulfill the needs of the course.

There are three objectives in the course "Power Processing II" which are: interesting undergraduate students in the power area of electrical engineering, providing the students with factual information and some experience relating to semiconductor power electronics, and to develop the students' ability to model physical problems. Student interest is fostered by the students' growing competence in the semiconductor power area, frequent classroom reference to the current engineering relevance of the types of problems being considered, and by making the problems and laboratory sessions as realistic as possible. Also, the area of semiconductor applications to power processing, the subject material of the course, is an area of great interest and expansion in the present day power industry and is therefore "relevant". Skill in modeling is encouraged in two ways. First, the problems the students are required to work are framed in terms of real circuit elements. The students must decide what idealizations can be made. Secondly, as the course progresses, the sophistication of the modeling required to solve the problems in a reasonable time increases, and hopefully the students' skill will increase as they work the gradually more sophisticated problems.

The author considers modeling the most important aspect of the course as reflected by the subject material and organization of this book. Two of the most valuable attributes of an engineer are his ability to assimilate new technology, and his ability to apply basic science and technology (new or old) to new problems. Without these attributes, the engineer is soon relegated to the position of a competent technician. One of the most, if not the most, powerful tools used to maintain these attributes is the engineer's skill in modeling physical problems; that is, to simplify the problem to the extreme so that the basic parameters and operations become obvious, and then to replace the necessary complexities until the model sufficiently approaches the real physical problem to give valid engineering answers. The author therefore feels that the gain in examining and modeling the problems in some detail far outweigh the disadvantage that less material (fewer circuits, problems, and applications) can be considered in the given time.

There are several reasons for stressing modeling in the particular course on semiconductor power processing. The primary objective in offering the course "Power Processing II" is to interest students in the power area of engineering. By incorporating the learning of a fundamental engineering skill (modeling) into the course, it may be possible to attract more of the "uncertain" students who may not want to commit themselves to a specific area of electrical engineering. The subject material may then provide sufficient challenge to interest these students in power engineering. Also, starting from the students undergraduate background in electronics, logic, and physics, the students actually experience the extension of their knowledge into an unfamiliar technological area (semiconductor power processing) using the tool of modeling as well as using modeling to solve complicated problems. And of course, even the simplest problems in power processing can only be solved by the straightforward application of Kirchoff's laws with utmost difficulty, further impressing upon the student the value of modeling as a problem solving tool.

The laboratory requires some special mention. The laboratory problems do not designate specific experiments to be performed by the students, nor is a "typical" formal laboratory report required of each student. While real-life situations sometimes require a "laboratory report", as in the testing and evaluation of an item or system, the most frequent use of an industrial laboratory is as an aid to finding the answer to a problem. The realistic laboratory problem associated with any problem is "What laboratory experiment should be done?" The student is given the choice of using the laboratory to gather data, confirm his theory, check assumptions, as an aid to understanding device or circuit operation, or any combination of these. The students are not permitted to enter the laboratory without a "plan" in which each student must identify a specific objective for the laboratory experiment, and a detailed plan to carry out the experiment. The students are graded on the basis of how effective their laboratory objective will be in enabling them to solve the problem, and whether their detailed plan has a reasonable assurance of enabling the students to accomplish their immediate objective. After the laboratory session, the students complete their assigned problem, presenting "an answer" which is backed up by laboratory experiment and data. This type of laboratory has proved much more interesting to the students and seems more in keeping with an engineering education than simply "verifying calculations" or "demonstrating effects."

End Quote

The courses were developed on the basis of determination of what should be in the course and the general approach to the method of presentation. Dr. T. W. Sze of the University of Pittsburgh and Mr. Alec H. B. Walker of Westinghouse Research and Development were active in the former activity and worked closely with Dr. Frank E. Acker and Dr. H. B. Hamilton throughout the life of the project. Class lecture notes were prepared, presented, revised, presented again and again revised based on the classroom experience and student reaction.

Student reaction has been favorable, especially in the power semiconductor work in Part II. It is the authors opinion that Part I, Electric Machinery Analysis, is excellent for the average and above average student. However, the below average student does have a difficult time with the material because of the mathematical rigor involved in the presentation.

Emphasis throughout is on modeling techniques. Students who develop the ability to model devices and systems seem to be well prepared for professional careers in both design and analysis. Further, the ability to model an entity insures that the student does have the capability for professional involvement in new and different activities and situations. It appears to mitigate against early technological obsolescence.

Two different individuals had direct responsibility for the two courses and the text material which evolved has different formats.

Part I, "Electric Machinery Analysis", consists of three bound booklets - the lecture notes, a problem manual, and a laboratory manual. The lecture notes emphasize modeling of a generalized machine and the resulting dynamic equations are then reduced (or modified) to portray specific types of real machines. Steady state behavior is obtained by an application of the Final Value Theorem to the Laplace transform of the describing equations. The laboratory work is an effort to relate the dynamic model studied in the classroom to the real world machine. The problems, from a variety of sources, serve to illustrate the lecture material.

Part II, "Modeling Power Processing Devices and Circuits" deals with power semiconductors - modeling their behavior, the thermal and other application problems and utilization of the theory to design and analyze a d.c. motor drive. Problems have been formulated and inserted in the text. Laboratory problems are also included at the end of each chapter which provide exercise in the laboratory application of classroom theory.

DISTRIBUTION

A set of the resulting project material, with the letter of transmittal detailed as Attachment #1 to this report has been forwarded to each Electrical Engineering Department in the U.S.A. and Canada.

In addition, requests for the material have been received from Mexico, Colombia, Chile and Greece. At this time, no firm decisions on publication and additional distribution have been formulated. Requests for suggestions have been sent out with the sets of project materials.

Notice of the material will be presented to the Power Engineering Education Committee of the Institute of Electrical and Electronic Engineers and to other appropriate groups.

CONCLUSION

The authors feel that the objectives of the project have been realized and that classroom tested text material, problems, and lab work suitable for a two semester course in power processing have evolved. We sincerely hope that sufficient interest will develop in other schools and universities to warrant further publication and distribution of the material.

The two sequence course will continue at the University of Pittsburgh. Part I as a required unit and Part II as an elective. Our faculty feel that this material is an essential ingredient in the educational process for electrical engineers.

PREFACE

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CHAPTER I - BASIC PRINCIPLES

1.1 INTRODUCTION: This text was prepared with a specific orientation which differs from the classical texts in electric machinery. It is oriented toward "power processing" and views the machine as an entity in a system which converts energy from one level, or form, to another level, or form. The emphasis is on analysis of machine behavior with only enough of the why of the behavior as is necessary to understand how behavior may be modified.

The system of units used is the rationalized mksc system. This system is generally accepted as the most useful for electrical engineering. In the mksc system many troublesome constants relating widely used quantities (such as power, torque, velocity, for example) are eliminated. It is not possible to eliminate all constants, however, The usage of the term 'rationalized' denotes an inclusion of the quantity 4π in such things as permeability of free space, etc. rather than usage as a constant for the relationship. Thus, Maxwell's equations can be written without use of the 4π . The English system of units is still in rather widespread usage in some areas - such as magnetic circuit design. However, it is being superceded by the mksc system.

It is also recommended that the student obtain and use a copy of the relatively inexpensive "Basic Tables in Electrical Engineering" (McGraw-Hill, 1965) by Korn. This reference contains much useful data on control systems, computer representation, motor characteristics, Laplace transform pairs, etc. plus other useful data and tables.

1.2 PERTURBATIONS: A very useful concept of the analysis of systems is that of the dynamic behavior of a system about a steady state operating point. The idea is that some small change, or perturbation, is introduced and the resulting system (or device) behavior analyzed. This concept can often be used to linearize a non-linear system and thus bring to bear on the analysis the powerful mathematical tools of operational mathematics that are available.

An interesting application of this concept is in the application of the principle of Conservation of Energy used to develop the relationships of the basic electro-mechanical conversion process. If we apply this principle and neglect electro-magnetic radiation (negligible at power frequencies) we can account for the various forms of energy present in the energy conversion process involving conversion from electrical to mechanical form, i.e., an electrical motor, as follows:

$$\begin{array}{c}\text{Energy input}\\ \text{from electrical}\\ \text{sources}\end{array} = \begin{array}{c}\text{mechanical}\\ \text{energy output}\end{array} + \begin{array}{c}\text{increase in}\\ \text{energy stored in the}\\ \text{coupling field}\end{array} + \begin{array}{c}\text{energy}\\ \text{converted}\\ \text{to heat}\end{array} \qquad (1\text{-}1)$$

For a generator, the electrical and mechanical energy terms would have negative values. The energy converted to heat results from i^2R electrical losses, friction and windage mechanical losses, and either magnetic or electric field losses. If these losses are grouped with the corresponding terms in (1-1), we have:

$$\begin{array}{lclcll} \text{Electrical} & & \text{mechanical} & & \text{energy increase} & \\ \text{Energy Input} & & \text{energy output} & & \text{stored in} & \\ \text{minus} & = & \text{plus friction} & + & \text{coupling field} & \text{(1-2)} \\ \text{Resistance} & & \text{and windage} & & \text{plus} & \\ \text{Losses} & & \text{losses} & & \text{field losses} & \end{array}$$

Denoting energy by W and incremental, or small changes, by ΔW and using an appropriate subscript to distinguish between electrical, mechanical and field energies, we can express (1-2) for the static, or steady state, case as:

$$W_{elec} = W_{mech} + W_{fld} \qquad (1\text{-}3)$$

If a small change, or perturbation, occurs it will effect the energy balance and we must rewrite (1-3) as:

$$(W_{elec} + \Delta W_{elec}) = (W_{mech} + \Delta W_{mech}) + (W_{fld} + \Delta W_{fld}) \qquad (1\text{-}4)$$

Subtracting (1-3) from (1-4) and allowing $\Delta W \rightarrow dW$ (the differential of W) yields:

$$dW_{elec} = dW_{mech} + dW_{fld} \qquad (1\text{-}5)$$

For small changes the losses are substantially constant and (1-5) can be applied to quantities only and neglecting losses. We will examine these quantities on a term by term basis and draw meaningful concusions.

The procedure of using only incremental changes in variables can also be used, in some instances, to linearize systems of non-linear differential equations if second and higher order terms are neglected. This method will be explored in detail in later chapters.

1.3 ELECTRICAL, MECHANICAL AND FIELD ENERGY: A device with one source of electrical input (a singly excited system) can be graphically portrayed as follows:

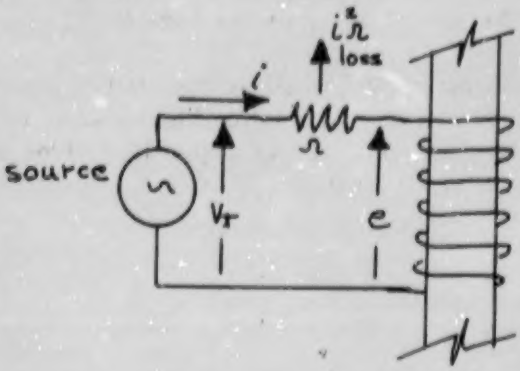


Figure I-1. Singly Excited Magnetic Circuit

where:

v_t = terminal voltage, instantaneous

i = current input, instantaneous

r = resistance

$e = v_t - ir$

Recall that instantaneous power, p, is given by the time rate of change of energy. Thus:

$$p = \frac{dW}{dt} \tag{I-6}$$

and:

$$dW_{input} = v_t i \, dt \tag{I-7}$$

$$dW_{losses} = i^2 r \, dt \tag{I-8}$$

Now the electrical energy input minus resistance losses is given by

$$dW_{elec} = v_t i\ dt - i^2 r\ dt = (v_t - ir) i\ dt = e\ i\ dt \qquad (1\text{-}9)$$

If electrical energy input is supplied from multiple sources, the total electrical energy input is the sum of terms of the form (1-9).

In Figure 1-1, a magnetic circuit is depicted as receiving an instantaneous power, ei. Faraday's law gives the relationship between the induced voltage, e, and the instantaneous flux linkages, λ. Lenz's law provides us with the directional relationship shown in Figure 1-1. That is:

$$e = \frac{d\lambda}{dt} \qquad (1\text{-}10)$$

where t is time, and the quantities are in the mks system. If the magnetic circuit has N turns, we can define an equivalent flux ϕ which is:

$$\phi = \frac{\lambda}{N} \qquad (1\text{-}11)$$

From (1-9), (1-10), and (1-11), we have

$$dW_{elec} = i\ d\lambda = N\ i\ d\phi = F\ d\phi \qquad (1\text{-}12)$$

where F = Ni is defined as magnetomotive force (mmf). Now mmf and flux are proportional to each other, the constant of proportionality is defined as the reluctance, R. Thus:

$$F = \phi R \qquad (1\text{-}13)$$

We are now in a position to evaluate total field energy W_{fld} as:

$$W_{fld} = \int_0^{\phi} \phi R\ d\phi = \frac{R\phi^2}{2} \qquad (1\text{-}14)$$

if the reluctance is constant.

Suppose the reluctance is not constant and that some value of flux, ϕ, exists. If we use the value of reluctance prevailing at that flux level, defined as R, we have a total energy stored in the field of:

$$W_{fld} = \frac{R\phi^2}{2} \tag{I-15}$$

Now, if the energy stored is changed by an amount ΔW_{fld}, flux and reluctance change by $\Delta\phi$ and ΔR respectively. We have then,

$$W_{fld} + \Delta W_{fld} = \frac{1}{2}(R + \Delta R)(\phi + \Delta\phi)^2 \tag{I-16}$$

or

$$W_{fld} + \Delta W_{fld} = \frac{1}{2}[R\phi^2 + \Delta R\phi^2 + 2R\phi\Delta\phi + 2\Delta R\phi\ \Delta\phi + R\Delta\phi^2 + \Delta R\Delta\phi^2] \tag{I-17}$$

Discarding second and higher order differentials and subtracting (I-15) from (I-17) yields,

$$\Delta W_{fld} = R\phi\Delta\phi + \frac{1}{2}\ \Delta R\phi^2 \tag{I-18}$$

As the incremental changes become smaller and smaller:

$$\Delta W_{fld} \rightarrow dW_{fld},\ \Delta\phi \rightarrow d\phi \text{ and } \Delta R \rightarrow dR$$

Thus,

$$dW_{fld} = R\phi d\phi + \frac{1}{2}\phi^2\ dR \tag{I-19}$$

We can evaluate mechanical energy by noting that:

$$W_{mech} = fx;\ \text{translational motion} \tag{I-20}$$

or

$$W_{mech} = T\theta;\ \text{rotational motion} \tag{I-21}$$

where

x = linear displacement
θ = rotational displacement
f = component of force in direction of x
T = component of torque in the direction of θ

for incremental changes:

$$W_{mech} + \Delta W_{mech} = (T + \Delta T)(\theta + \Delta\theta)$$

$$= T\theta + \theta\Delta T + T\Delta\theta + \Delta T\Delta\theta \qquad (1\text{-}22)$$

Subtracting (1-21) from (1-22), neglecting the second order effect and passing to the limit as the incremental change approaches zero yields:

$$dW_{mech} = Td\theta + \theta dT \qquad (1\text{-}23)$$

Further, for very small changes in θ, we can assume that the torque (or the force, if appropriate) remains constant over θ and we can write:

$$dW_{mech} = Td\theta \text{ or } T = fdx \qquad (1\text{-}24)$$

1.4 TORQUE AND FORCE: We can now substitute (1-12, -19 and -24) into (1-5), yielding:

$$Fd\phi = Td\theta + R\phi d\phi + \frac{1}{2}\phi^2 dR \qquad (1\text{-}25)$$

Noting that $F = \phi R$, (1-25) becomes:

$$T = -\frac{1}{2}\phi^2 \frac{dR}{d\theta} \text{ (rotational)} \qquad (1\text{-}26)$$

or

$$f = -\frac{1}{2}\phi^2 \frac{dR}{dx} \text{ (translational)} \qquad (1\text{-}27)$$

This says that the force involved in a small change of configuration is proportional to the change of reluctance with respect to the displacement involved and the square of the flux in the system. The negative sign merely indicates that the force, or torque, is in a direction to reduce the reluctance.

Let us reexamine the stored energy in the field, (1-15). If we had merely taken the partial derivative with respect to R, ϕ, we would have obtained

$$dW'_{fld} = \frac{\phi^2}{2} dR + \phi R d\phi \qquad (1\text{-}28)$$

which is the same as (I-19). This will allow us to develop a generalization for torque (or force) by considering the two situations, a) constant flux or b) constant mmf. To do so, rewrite (I-15) as:

$$W_{fld} = \frac{1}{2}\ R\phi^2 = \frac{F\phi}{2} \tag{I-29}$$

a) For ϕ = constant with varying F;

$$dW_{fld} = \frac{\phi}{2}\ dF \tag{I-30}$$

from (I-12), -24 and -30) in (I-5), we have:

$$Fd\phi = Td0 + \frac{\theta}{2}\ dF \tag{I-31}$$

but, $d\phi = 0$. Therefore:

$$T = -\frac{\phi}{2}\frac{dF}{d} \tag{I-32}$$

b) For F = constant with varying ϕ;

$$dW_{fld} = \frac{F}{2}\ d\phi \tag{I-33}$$

and, substituting into the energy balance equation yields

$$Fd\phi = Td\theta + \frac{F}{2}\ d\phi \tag{I-34}$$

or

$$T = +\frac{F}{2}\frac{d\phi}{d\theta} \tag{I-35}$$

If we note that, from (I-29)

$$\left.\frac{\partial W_{fld}}{\partial \theta}\right|_{\phi = \text{constant}} = \frac{\phi}{2}\frac{dF}{d\theta} \tag{I-36}$$

and

$$\left.\frac{\partial W_{fld}}{\partial \theta}\right|_{F = \text{constant}} = \frac{F}{2}\ \frac{d\phi}{d\theta} \tag{I-37}$$

we can rewrite (I-32) and (I-35) as

$$T = - \frac{\partial W_{fld}}{\partial \theta}\bigg|_{\theta \,=\, \text{constant}} \qquad \text{(I-38)}$$

and

$$T = + \frac{\partial W_{fld}}{\partial \theta}\bigg|_{F \,=\, \text{constant}} \qquad \text{(I-39)}$$

Equation (I-38) and (I-39) are extremely useful to us because they enable us to express force or torque as functions of variables such as reluctance, flux linkages, current, inductance, etc. For example:

$$W_{fld} = \frac{F\phi}{2} = (\frac{Ni}{2})(\frac{Li}{N}) = \frac{1}{2} L\, i^2 \qquad \text{(I-40)}$$

where

$$L = \text{magnetic inductance } (L = \frac{N\phi}{I})$$

For constant mmf, i = constant and, in terms of force:

$$f = \frac{\partial W_{fld}}{\partial x}\bigg|_{F \,=\, \text{constant}} = \frac{\partial}{\partial x} (\frac{1}{2} L\, i^2) = \frac{i^2}{2} \frac{dL}{dx} \qquad \text{(I-41)}$$

or, for another example

$$W_{fld} = \frac{1}{2} F\phi = \frac{F}{2} (FP) = \frac{F^2 P}{2} \qquad \text{(I-42)}$$

where

$$P = \text{magnetic circuit permeance, } (P = \frac{1}{R}).$$

For constant mmf, in terms of force

$$f = \frac{\partial W_{fld}}{\partial x}\bigg|_{F \,=\, \text{constant}} = \frac{\partial}{\partial x} (\frac{F^2 P}{2}) = \frac{F^2}{2} \frac{\partial P}{\partial x} \qquad \text{(I-43)}$$

and so on.

From (I-41) we can draw a very important conclusion concerning electric machines. If we note that, at constant mmf, $Li = \lambda$ and $i dL = d\lambda$, we have (using torque rather than force)

$$T = \frac{i}{2}\frac{d\lambda}{d\theta} \qquad (1\text{-}44)$$

Whereas the voltage inducing phenomena is related to the time rate of change of flux linkages, the torque producing mechanism is based on the angular rate of change of flux linkage.

To summarize, the torque or forces, act as follows:

1. To decrease the stored energy at constant flux.
2. To increase the stored energy at constant mmf.
3. To decrease the reluctance (or increase the permeance).
4. To increase the inductance.

Actually, the torques and forces are on the iron member itself in an iron cored magnetic circuit. The presence of a winding of some sort merely serves to establish the field. Thus windings in slots on a rotor are not themselves subjected to the torques exerted on the iron. The same force that tends to reduce reluctance also produces a stress within the iron itself. This stress sets up strains within the iron circuit which causes a change in shape. This general phenomena is known as magnetostriction. If the magnetic field is an alternating field, the periodic change in shape of the magnetic structure causes pressure waves in the surrounding media. The pressure waves cause audible noise in some instances.

It is sometimes useful to analyze problems using energy density, or field energy per unit volume. Thus

$$w_{fld} = \frac{W_{fld}}{vol} = \frac{F\phi}{2(\ell A)} = \frac{1}{2}\left(\frac{Ni}{\ell}\right)\left(\frac{\phi}{A}\right) \qquad (1\text{-}45)$$

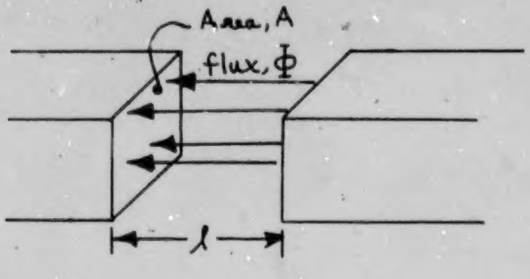
where ℓ and A are length and cross sectional area of the magnetic circuit under consideration. Since: $\frac{Ni}{\ell} = H$, magnetic field intensity,

and: $\frac{\phi}{A} = B$, flux density

$$w_{fld} = \frac{1}{2} BH = \frac{1}{2}\frac{B^2}{\mu} \qquad (1\text{-}46)$$

where μ is the permeability of the circuit and $B = \mu H$.

As an example of the usefulness of this approach, we will drive an expression for the force of attraction between two parallel faces of high permeability permanent iron magnets with flux passing between them as shown in Figure 1-2.



$B = \phi A$

μ_o = permeability of free space
$= 4\pi \times 10^{-7}$

μ = permeability of iron

$\mu >> \mu_o$

Figure 1-2. Air gap between two Magnets

We will neglect flux fringing. The permeability of the iron is very high and that of the air between the faces is relatively low. Therefore the energy density is much higher in the air gap than in the iron. Therefore we will consider only the energy density in the air gap.

$$w_{fld} = \frac{1}{2}\frac{B^2}{\mu_o} \tag{1-47}$$

$$W_{fld} = \frac{1}{2}\frac{B^2}{\mu_o} A\ell \tag{1-48}$$

The mmf is constant since these are permanent magnets. The force involved here is in a direction to reduce the reluctance, i.e., a force of attraction. An incremental change can only be one where ℓ is reduced by an infinitesimal amount.

$$f = \frac{\partial W_{fld}}{\partial x} = \frac{1}{2}\frac{B^2 A}{\mu_o}\frac{d\ell}{dx} \tag{1-49}$$

but

$$dx = -\,d\ell$$

$$\therefore \quad f = -\frac{1}{2}\frac{B^2 A}{\mu_o} \tag{1-50}$$

where the negative sign indicates force is in the direction to reduce ℓ.

1.5 RELATIONSHIPS IN MULTIPLE EXCITED SYSTEMS: We will now extend our analysis to cover an energy conversion device with multiple electrical inputs. (A very simplified version). Figure 1-3 portrays such a device. One winding is on the stationary member, the other on the moveable member. Each winding has a self inductance, L_{11} and L_{22}, respectively. In addition, a mutual inductance, L_{12}, represents the magnetic coupling between the two windings. Each of these inductances is a function of the angular displacement of the moveable member (or rotor).

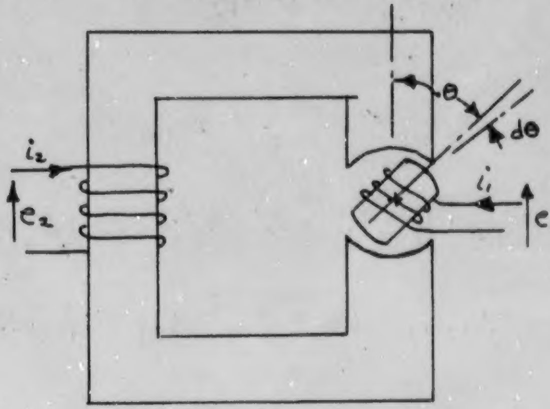


Figure 1-3. A Multiple Excited Electromechanical Energy Converter

The analysis will be based on an incremental change in angular displacement, $d\theta$, as used in the singly excited system analysis. However, we will utilize the inductances to express the various electrical quantities. Recall that flux linkages can be expressed as the product of current and inductance.

Thus, from (1-12):

$$dW_{elec} = i_1 d\lambda_1 + i_2 d\lambda_2 \qquad (1\text{-}51)$$

$$= i_1 d(i_1 L_{11} + i_2 L_{12}) + i_2 d(i_2 L_{22} + i_1 L_{12})$$

$$= i_1^2 dL_{11} + i_1 L_{11} di_1 + i_1 i_2 dL_{12} + i_1 L_{12} di_2 + i_2^2 dL_{22} +$$

$$+ i_2 L_{22} di_2 + i_2 i_1 dL_{12} + i_2 L_{12} di_1 \qquad (1\text{-}52)$$

Also, recall that, in terms of inductance:

$$W_{fld} = \frac{1}{2} L_{11} i_1^2 + \frac{1}{2} L_{22} i_2^2 + L_{12} i_1 i_2 \qquad (1\text{-}53)$$

and, therefore:

$$\begin{aligned} dW_{fld} = {} & \frac{1}{2} i_1^2 dL_{11} + i_1 L_{11} di_1 + \frac{1}{2} i_2^2 dL_{22} + i_2 L_{22} di_2 + L_{12} i_1 di_2 + \\ & + L_{12} i_2 di_1 + i_1 i_2 d\, L_{12} \end{aligned} \qquad (1\text{-}54)$$

from:

$$dW_{elec} = dW_{mech} + dW_{fld} \qquad (1\text{-}5)$$

$$\begin{aligned} & i_1^2 dL_{11} + i_1 L_{11} di_1 + i_1 i_2 dL_{12} + i_1 L_{12} di_2 + i_2^2 dL_{22} + i_2 L_{22} di_2 \\ & + i_2 i_1 dL_{12} + i_2 L_{12} di_1 = T d\theta + \frac{1}{2} i_1^2\, dL_{11} + i_1 L_{11} di_1 + \frac{1}{2} i_2^2 dL_{22} \\ & + i_2 L_{22} di_2 + L_{12} i_1 di_2 + L_{12} i_2 di_1 + i_1 i_2 dL_{12} \end{aligned} \qquad (1\text{-}55)$$

we have:

$$T = \frac{1}{2} i_1^2 \frac{dL_{11}}{d\theta} + \frac{1}{2} i_2^2 \frac{dL_{22}}{d\theta} + i_1 i_2 \frac{dL_{12}}{d\theta} \qquad (1\text{-}56)$$

Note that the torque (or force, if θ is replaced by x in a translational system) is a result of changes of inductance with angular displacement. There may well be changes of current due to electrical transients but these changes are not effective in the production of torque.

Further insight into this can be obtained by examination of one of the induced voltages. For example:

$$\begin{aligned} e_1 & = \frac{d}{dt} \left[L_{11} i_1 + L_{12} i_2\right] \\ & = L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} + \left(i_1 \frac{dL_{11}}{d\theta} + i_2 \frac{dL_{22}}{d\theta}\right) \frac{d\theta}{dt} \end{aligned} \qquad (1\text{-}58)$$

The first two terms on the right hand side are "transformer" voltages which occur in any magnetic circuit when the flux linkages change with respect to time. The last two terms on the right are "speed" or rotational voltages, since speed = $\frac{d\theta}{dt}$.

Note that only $\frac{dL}{d\theta}$ terms appear in (1-56), the expression for torque. Thus, we conclude that speed voltages and not transformer voltages are active in the energy conversion process. Our physical reasoning verifies this because we know that (neglecting losses) no energy conversion takes place in a transformer!

1.6 ELECTROSTATIC RELATIONSHIPS: The same energy balance analysis can be applied to an electrostatic energy converter system. That is:

$$dW_{elec} = dW_{mech} + dW_{fld} \tag{1-5}$$

Consider a parallel plate capacitor excited from a single source of potential e.

$$dW_{elec} = ei\ dt = e\ dq \tag{1-59}$$

where:

$$dq = \text{differential charge} = idt$$

Further, stored energy in the electrostatic field is given by:

$$W_{fld} = \frac{1}{2} Ce^2 \tag{1-60}$$

and

$$dW_{fld} = \frac{1}{2} e^2\ dC + eC\ de \tag{1-61}$$

for incremental changes.

Applying the energy balance and using the relationships (1-20), (1-59), and (1-61) yields:

$$e\ dq = \frac{1}{2} e^2\ dC + eC\ de + f\ dx \tag{1-62}$$

where:

f = force on the capacitor plates

dx = incremental displacement that would change C recall that:

$$q = eC$$

therefore;

$$dq = e\ dC + C\ de \tag{1-63}$$

for incremental changes, using the same reasoning as applied in our study of electromagnetic circuits. Substituting (1-63) for dq into the left hand side of (1-62) yields:

$$e^2\ dC + eC\ de = \frac{1}{2} e^2\ dC + eC\ de + f\ dx \tag{1-64}$$

or:

$$f = +\frac{1}{2} e^2 \frac{dC}{dx} \tag{1-65}$$

The force on the plates of a capacitor act in a direction to increase the capacitance. An analogous expression for torque also exists, of course.

1.7 ELECTROMAGNETIC VS. ELECTROSTATIC ENERGY CONVERTERS:

From the previous analysis, it should be apparent that the energy conversion process is dependent upon the existence of the electric or magnetic field as a coupling media. For example, in an electric motor energy flows in from the electrical sources and is transferred from a stationary member to a rotating member where it appears as mechanical energy. Since this energy must pass through an air gap of some configuration, it can be reasoned that the air gap must be capable of storing this energy - even if it is energy in continuous flow. We have shown that energy conversion is possible in either an electromagnetic or electrostatic device.

Many are familiar with the electromagnetic type of converter but few have familiarity with the electrostatic type. Why is this? Why is one more widely used than the other? It is because of the relative, maximum, energy densitites possible. That is to say, for the same volume of air gap one can store much more energy than the other. What are the energy storage capabilities? Recall, from (1-46)

$$w_{fld\ magnetic} = \frac{1}{2} \frac{B^2}{\mu_o} \tag{1-46}$$

For a capacitor

$$w_{fld\ electric} = \frac{1}{2} \frac{Ce^2}{vol} = \frac{1}{2} \frac{q}{A} \left(\frac{e}{\ell}\right) = \frac{1}{2} DE \tag{1-66}$$

$$= \frac{1}{2} \varepsilon E^2 \tag{1-67}$$

where:

E = electric field intensity, volts/meter

D = electrostatic flux density, coulombs/meter^2

ε = permittivity of the dielectric medium

Flux densities of 1 weber/meter^2 are quite common whereas the maximum electric field intensity of air is 3×10^6 volts/meter (with failure of the air as an insulator, or dielectric). The permeability of air, μ_o, is $4\pi \times 10^{-7}$. The permittivity of air is 8.85×10^{-12}. Thus, from (I-46) and (I-67)

$$\frac{w_{\text{fld magnetic, max.}}}{w_{\text{fld electric, max.}}} = \frac{B^2}{\mu_o \varepsilon E^2} \tag{I-68}$$

$$= \frac{1^2}{4\pi \times 10^{-7} \times 8.85 \times 10^{-12} \times 9 \times 10^{12}}$$

$$= 10{,}000$$

It would require 10^4 times as much volume to store the same energy in an electrostatic field in air as it would be to store electromagnetic energy in air. Also, it should be noted that high power electromagnetic devices involve high currents at the necessary driving voltage, whereas electrostatic energy converters involve high voltage, if their power or energy rating is to be substantial. We provide conductor size for high currents and insulation for high voltage. The mechanical design problems are thus quite different!

CHAPTER II - ELEMENTARY CONCEPTS OF TORQUE AND GENERATED VOLTAGE

II.1 THE D.C. MACHINE: The simplest possible visualization of generated voltage in an electric machine (whether generator or motor) can be gained from the study of an elementary d.c. machine. A simplified version of a d.c. machine is shown in Figure II-1

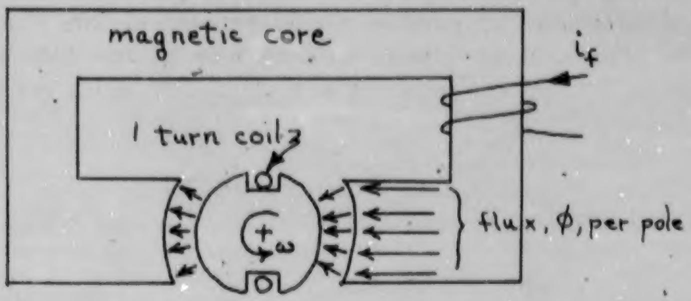


Figure II-1. Elementary D.C. Machine

Our elementary d.c. machine has an exciting winding, usually called the "field" winding on the stationary member of the machine (usually referred to as the "stator"). This winding establishes the flux, ϕ , which passes from the "pole", across the air gap, through the rotating member, or "rotor", through the air gap again and back into the stator pole. In an actual machine, the winding on the rotor would be distributed around the periphery of the rotor with the various coils connected in some specific series-parallel combination to yield the desired volt-ampere rating of the machine. In the particular simplified elementary machine to be analyzed here, we will assume only one coil, consisting of one turn only. As the coil rotates, the flux linking it changes with time. Not shown is the mechanical switching device (commutator) which is essential to the machine if a unidirectional voltage is to exist at the collecting mechanism (brushes) connected to the rotating coil. However, we can examine the voltage generating mechanism. When the coil is rotated 1/4 of the revolution ($\frac{\pi}{2}$ radians), the flux linking the coil changes from ϕ to zero. If the coil is rotating at ω radians/sec, the time to rotate $\frac{\pi}{2}$ radians is, $\Delta t = \frac{\pi}{2\omega}$. If the machine had p poles, the rotor would rotate through $\frac{\pi}{2}$ electrical radians. This would be $(\frac{\pi}{2})/p/2$ mechanical radians. For the p pole machine, then, $\Delta t = \frac{\pi}{p\omega}$.

The voltage induced in the one turn coil is given by Faraday's law as:

$$e = \frac{\Delta\phi}{\Delta t} = \frac{\phi - 0}{\frac{\pi}{p\omega}} = \frac{p\omega\phi}{\pi} \qquad \text{(II-1)}$$

Now, consider a more complicated rotor winding. Assume there are a total of Z inductors on the rotor forming $\frac{Z}{2}$ one turn coils. If the winding is arranged so that there are "a" parallel paths through the rotor winding (the current to the rotor enters one collector, divides into "a" parallel paths, recombines into the original current and exits through the other collector) then there must be $\frac{Z}{2a}$ one turn coils in series. The total voltage across the collectors is the sum of all the series voltages, i.e.;

$$e = \left(\frac{Z}{2a}\right)\left(\frac{p\phi\omega}{\pi}\right) = \frac{Z\phi p}{a}\left(\frac{\omega}{2\pi}\right) \qquad \text{(II-2)}$$

If: n = revolutions/minute

$\frac{n}{60}$ = revolutions/second

Note, in (II-2), that $\frac{\omega}{2\pi}$ = revolutions/second also. If the bracked term, involving radians/second, in (II-2) is replaced by its equivalent in terms of revolutions/second, we have:

$$e = \frac{Z\phi pn}{a\ 60} \qquad \text{(II-3)}$$

If a current, i, flows in the winding, electric power is associated with the machine. If it is a motor, the current flows against the induced voltage and is absorbed. The absorption of this power corresponds to the power being converted to mechanical form. If the current flows in the direction of the induced voltage, power is supplied to an external load and the machine acts as a generator. This source of power can only come from mechanical shaft input power and the device behaves as a generator.

If: T = torque, in newton meters

P = power, in watts

$$P = \omega T = ei \qquad \text{(II-4)}$$

Equation (II-4) can be solved for torque and the relationship in (II-2) utilized to yield:

$$T = \frac{ei}{\omega} = \left(\frac{Zp}{2\pi a}\right) \phi i = k\phi i \qquad \text{(II-5)}$$

where

$$k = \frac{Zp}{2\pi a}$$

It should be noted that k is a constant whose value is determined by the design of the machine. If the value of k is substituted in (II-2), we have:

$$e = \left(\frac{Zp}{2\pi a}\right) \phi\omega = k\phi\omega \qquad \text{(II-6)}$$

In the d.c. machine considered here, both developed torque and induced voltage, sometimes called "counter electromotive force" (cemf) or "back" emf (because of its polarity relative to that of the external circuit), are proportional to the flux in the air gap resulting mainly from the excitation or field winding. The torque is also proportional to the magnitude of current through the rotor (also called the armature). Thus we could generalize by noting that the torque is the result of interaction between a magnetic field and an mmf. The voltage induced is proportional to the angular velocity of the rotor inductors with respect to the magnetic field flux. In each case, the constant of proportionality is the same - but only in the mks system of units. That this is so one would expect from the consistency of the $F = Bi\ell$, $e = B\ell v$ relationships applicable to individual conductors. The device analyzed here has merely been a mechanical configuration to utilize the fundamental principles and to make use of multiple conductors with an appropriate series-parallel connection.

II.2 THE ALTERNATING CURRENT MACHINE: An elementary alternating current machine is shown in Figure II-2. Consider that the one turn coil on the stator is stationary and that the flux density (constant in magnitude, sinusoidal in space distribution, and directed along the axis of the rotating member) rotates at angular velocity ω. The flux is everywhere normal to the surface of the magnetic circuit. The flux density has a linear velocity tangential to the radius and each side of the one turn coil has a voltage induced in it of magnitude:

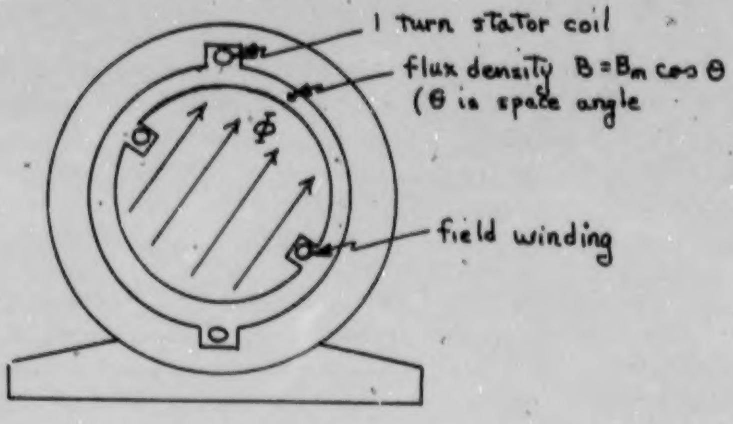
$$e = B(\theta)\ell v \qquad \text{(II-7)}$$

where

$B(\theta)$ = flux density, webers/meter^2

ℓ = length of the coil side in meters

v = relative linear velocity, meters/sec between stator coil and the flux.



1 turn stator coil
flux density $B = B_m \cos\theta$
(θ is space angle
Φ
field winding

Figure II-2. Elementary One Turn A.C. Generator (or Alternator)

Under the assumption of sinusoidal distribution in space of the flux density, the flux density can be expressed as

$$B(\theta) = B_m \cos\theta \qquad \text{(II-8}$$

where

B_m = maximum value of flux density

θ = space angle measured from the axis of the rotor

Since there are two sides in the one turn coil, the total induced voltage in the coil is:

$$e = 2B(\theta)\,\ell\,v = 2\,B_m\,\ell v\,\cos\theta \qquad \text{(II-9)}$$

If there are N turns on the stator coil, and the turns are connected in series, the voltage in the coil is given by

$$e = 2N\,B(\theta)\,\ell v \qquad \text{(II-10)}$$

Since:

$$\theta = \omega t \text{ and } \quad v = r\omega$$

where:

t = time, in seconds
r = radius of the rotor, in meters

The expression for induced voltage, (II-10) becomes:

$$e = 2N \ell r \omega B_m \cos \omega t \qquad \text{(II-11)}$$

or:

$$e = E_{max} \cos \omega t \qquad \text{(II-12)}$$

Recall that flux, ϕ, is related to flux density by:

$$\phi = \int_A B dA \qquad \text{(II-13)}$$

where A is the area normal to the flux density, B.

The area element, for the configuration in Figure II-2 is:

$$dA = \ell r \, d\theta \qquad \text{(II-14)}$$

Note that the incremental angle $d\theta$ is actual, or so called "mechanical", radians whereas the flux density distribution is in electrical radians. That is, in passing through two magnetic poles, 2π electrical radians are traversed - which may not be the same number of mechanical radians in the same angle. If the mechanical angle is multiplied by the number of pairs of poles, it is converted to an equivalent electrical angle. Thus,

Mechanical	Electrical	
$d\theta$	$\frac{d\theta}{p/2}$	(II-15)

In terms of electrical radians, (II-14) becomes:

$$dA = \frac{\ell r \, d\theta}{p/2} \qquad \text{(II-16)}$$

In order to determine the flux, per pole, we can evaluate (II-13) utilizing (II-8) and (II-16) yielding:

$$\phi = \int_{-\pi/2}^{\pi/2} \frac{\ell r\ B_m}{p/2} \cos\theta\ d\theta = \frac{2\ \ell r\ B_m}{p/2} \qquad \text{(II-17)}$$

From (II-11) and (II-12):

$$E_{max} = 2N\ \ell\ r\ \omega\ B_m \qquad \text{(II-18)}$$

From (II-17):

$$2\ \ell\ r\ B_m = \frac{p}{2}\phi \qquad \text{(II-19)}$$

Substituting (II-19) into (II-18) yields:

$$E_{max} = N\ \omega\ \phi\ \frac{p}{2} \qquad \text{(II-20)}$$

Note that rotation of the rotor through an angle which includes a pair of poles will produce 1 cycle of voltage variation. A complete revolution, through p/2 pairs of poles will produce p/2 cycles. Now, $\omega/2\pi$ revolutions/second through p/2 pairs of poles will produce:

$$\left(\frac{\omega}{2\pi}\right) \frac{p}{2} \text{ cycles in each second} \qquad \text{(II-21)}$$

Thus, the cyclic frequency of variation of voltage, f, is given by:

$$f = \frac{p\omega}{4\pi} \qquad \text{(II-22)}$$

and,

$$\frac{\omega p}{2} = 2\pi f \qquad \text{(II-23)}$$

Since the induced voltage is sinusoidal, E_{max} from (II-20) can be expressed in terms of root-mean-square (rms) value as:

$$E_{rms} = \frac{E_{max}}{\sqrt{2}} = \frac{N\phi}{\sqrt{2}}\left(\frac{\omega p}{2}\right) \qquad \text{(II-24)}$$

From (II-23),

$$E_{rms} = \frac{N\phi}{\sqrt{2}}\left(\frac{\omega p}{2}\right) = \left(\frac{2\pi}{\sqrt{2}}\right) N\phi f = 4.44\ N\phi f \qquad \text{(II-25)}$$

An actual machine has windings distributed over the entire periphery of the stator and the various inductors connected in some series-parallel arrangement. The number of turns, N, in (II-25) then becomes the number of turns connected in series. However the induced voltages, at any instant, in the various turns are not equal because of the distribution of flux density. Also, for purposes of suppressing undesired harmonics in the induced voltage some windings have coils whose sides do not span an exact pole pitch. Thus, the actual voltage is less than the value calculated by (II-25) in an actual machine. To account for the factors (II-25) is multiplied by a constant called the winding constant, whose value is determined by the specific winding configurations.

The relationship given in (II-25) applies to the induced voltage in any coil - such as a transformer, for example - which has sinusoidal voltages and fluxes present.

Consider the N turn coil and core in Figure II-3.

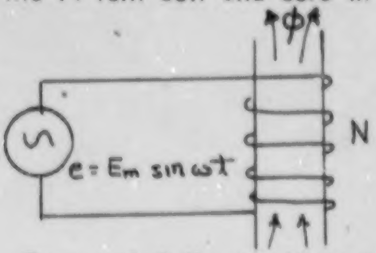


Figure II-3. Core and Coil with Sinusoidal Excitation

Neglecting winding resistance, a voltage is induced in the coil equal and opposite to e. From Faraday's Law,

$$e = E_m \sin \omega t = N \frac{d\phi}{dt} \qquad \text{(II-26)}$$

where the subscript m denotes maximum value.

If the flux is sinusoidal, it can be expressed as

$$\phi = \phi_m \cos \omega t \qquad \text{(II-27)}$$

and the induced voltage, which must equal the applied voltage, (if the resistance is zero) is given by:

$$e = N \phi_m \omega \sin \omega t = E_m \sin \omega t \qquad \text{(II-28)}$$

From (II-28)

$$E_m = N\phi_m \omega = N\,\phi_m\,(2\pi f) \tag{II-29}$$

The rms value of induced voltage can be then determined as:

$$E_{rms} = \frac{E_m}{\sqrt{2}} = \frac{2\,\pi}{\sqrt{2}}\;N\phi f \approx 4.44\;N\phi f \tag{II-30}$$

The production of torque is an elementary d.c. machine was developed in the previous section, equation (II-5). We will now examine the torque resulting from a different mechanical configuration - one which is used for machines supplied from alternating current (A.C.) sources. Commercial A.C. sources have periodic variations and thus consist of predominately a fundamental sinusoid with the possibility existing of additional sinusoids that are multiples of the fundamentals; i.e., harmonics.

The configuration we will analyze is shown in Figure II-4 and consists of two windings mounted on concentric magnetic cylinders.

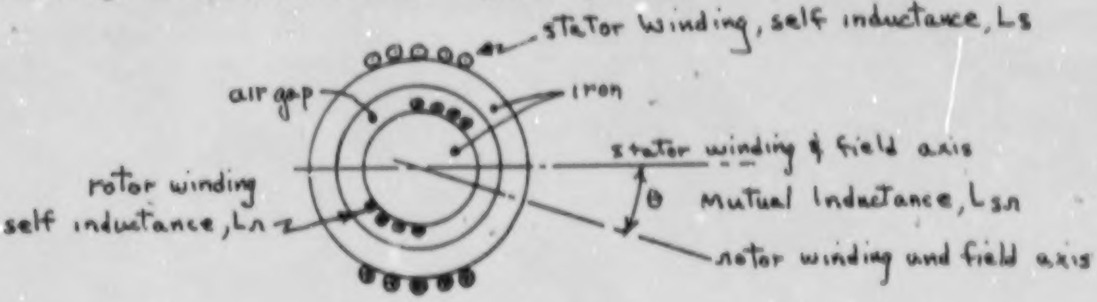
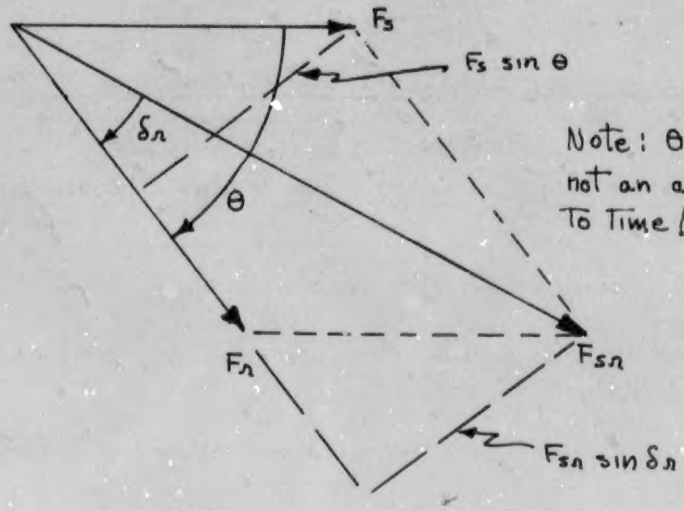

stator winding, self inductance, L_s
air gap
iron
stator winding & field axis
rotor winding
self inductance, L_r
θ
Mutual Inductance, L_{sr}
rotor winding and field axis

Figure II-4. Cylindrical Rotor Model

This is a simplified version of what could be either a stationary or rotating device. For example, the stator winding (on coil) as shown might be a fictitious winding to represent the source of an mmf which is rotating. The only stipulation, if the windings are rotating, is that the axis of the two windings be stationary with respect to each other under steady state (i.e. constant torque load) conditions. This is essential if average torque is to be produced.

Since the structure is concentric, the self inductances are constant. The mutual inductance is a function of the angle between the two axis, θ. We will assume that the variation of mutlal inductance is sinusoidal and that the mmfs of each winding are sinusoidally distributed, also. Further, we will assume that the iron is infinitely permeable. Although Figure II-4 is for a 2 pole device, our derivation will be for a p pole device. Only two poles are shown for simplicity. Each winding will have an mmf, F and F_r, and the net mmf in the air gap, F_{sr}, will be the resultant of the two mmfs[5]. Since the mmfs are sinusoidal, their resultant will also be sinusoidal and can be obtained by using phasors to represent the peak mmfs and combining them using the Law of Cosines. Figure II-5 depicts the phasor relationship between the rotor mmf, F_r, the stator mmf, F_s, and the air gap mmf, F_{sr}.



Note: θ is a space angle, not an angle corresponding To Time!

Figure II-5. Phasor Diagram of mmfs

From Figure II-5, applying the Law of Cosines:

$$F_{sr}^2 = F_s^2 + F_r^2 + 2 F_s F_r \cos \theta \tag{II-31}$$

We will define an angle, δ_r, to be the angle (in electrical radians between the axis of the rotor and the axis of the net air gap mmf. Geometric and trigonometric considerations indicate that:

$$F_s \sin \theta = F_{sr} \sin \delta_r \tag{II-32}$$

In order to obtain the torque produced, we will use the derived relationship, for ϕ = constant, from Chapter I, i.e.,:

$$T = - \frac{p}{2} \frac{\partial W_{fld}}{\partial \theta} \tag{I-38}$$

The factor p/2 takes into account our p pole machine.

To obtain W_{fld}, we will use average energy density times gap volume. If the diameter of the air gap mid point is d, the gap length is g, and the axial length of the gap is ℓ, then,

$$\text{total air gap volume} = \pi d \, \ell g \tag{II-33}$$

From (I-46), the energy density is:

$$w_{fld} = \frac{1}{2}\frac{B^2}{\mu_o} = \frac{1}{2}\mu_o H^2 \qquad \text{(I -46)}$$

This is energy density at any point in the air gap. The average energy density is given by:

$$w_{fld} = \frac{\mu_o}{2}\,(H^2)\text{ average} \qquad \text{(II-34)}$$

Since flux density is sinusoidally distributed in space, the corresponding mmfs are also sinusoidally distributed.

If F_{sr} is the peak value of air gap mmf, the peak value of magnetizing force, H_p, is given by:

$$H_p = \frac{F_{sr}}{g} \qquad \text{(II-35)}$$

and:

$$H = H_p \cos\theta \qquad \text{(II-36)}$$

The average value of the square of the H can be determined as:

$$(H^2)\text{ ave.} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} H_p^{\,2}\cos^2\theta\, d\theta = \frac{H_p^{\,2}}{2} \qquad \text{(II-37)}$$

From (II-33), (II-34) and (II-37), total energy in the air gap is:

$$W_{fld} = (w_{fld})(\text{volume}) = \frac{\mu_o H_p^{\,2}}{4}\,\pi d\ell g \qquad \text{(II-38)}$$

Substituting (II-35) for H_p yields:

$$W_{fld} = \mu_o \frac{\pi d\ell}{4g} F_{sr}^{\,2} \qquad \text{(II-39)}$$

From (I-38):

$$T = -\frac{p}{2}\frac{\partial}{\partial\theta}\left(\mu_o \frac{\pi d\ell}{4g} F_{sr}^{\,2}\right) \qquad \text{(II-40)}$$

Using the value of $F_{sr}^{\,2}$ from (II-31) in (II-40) yields:

$$T = -\frac{p}{4g}\,\mu_o \pi d\ell\, F_s F_r \sin\theta \qquad \text{(II-41)}$$

From (II-32), F_s can be replaced in (II-42) yielding:

$$T = -\frac{p}{4g}\mu_o \pi d\ell F_{sr} F_r \sin\delta_r \qquad \text{(II-42)}$$

From (II-35):

$$F_{sr} = g H_p = g \frac{B_{sr}}{\mu_o} \qquad \text{(II-43)}$$

where B_{sr} is the peak value of net flux density sinusoidally distributed in the air gap and that, for sinusoidal distribution:

$$B_{ave} = \frac{2 B_{sr}}{\pi} \qquad \text{(II-44)}$$

therefore, utilizing (II-44), (II-43) becomes:

$$F_{sr} = \frac{\pi}{2} \frac{g}{\mu_o} B_{ave} \qquad \text{(II-45)}$$

Note also that:

$$\pi\ell \frac{d}{2} = (\frac{p}{2}) \text{ (area under each pole)} \qquad \text{(II-46)}$$

Substituing (II-45) and (II-46) into (II-42) yields

$$T = -(\frac{p}{2})^2 \frac{\pi}{2} \text{(area under each pole) } B_{ave} F_r \sin\delta_r \qquad \text{(II-47)}$$

Now,

$$(B_{ave})\text{(area under ach pole)} = \phi_{sr} \qquad \text{(II-48)}$$

where

ϕ_{sr} = flux per pole in the air gap

Then, (II-47) becomes:

$$T = -(\frac{p}{2})^2 \frac{\pi}{2} \phi_{sr} F_r \sin\delta_r \qquad \text{(II-49)}$$

This equation expresses torque as the interaction between a field and an mmf or between two fields. It should be noted that no torque (or power) is possible without a finite angle, δ_r. It is very appropriately referred to as the "Power" angle or "torque" angle and will appear again in every device which we study. For the d.c. machine the angle is fixed at $\frac{\pi}{2}$ radians by the geometry of the machine. Thus, although it does not appear in the equation for torque in the d.c. machine it would be there if we had an angle other than $\pi/2$ radians between the flux density, corresponding to ϕ, and the mmf, corresponding to rotor current.

CHAPTER III - THE GENERALIZED MACHINE

III.1 THE BASIC MACHINE: All types of electrical machines have many features in common and Figure *III-1* depicts these features.

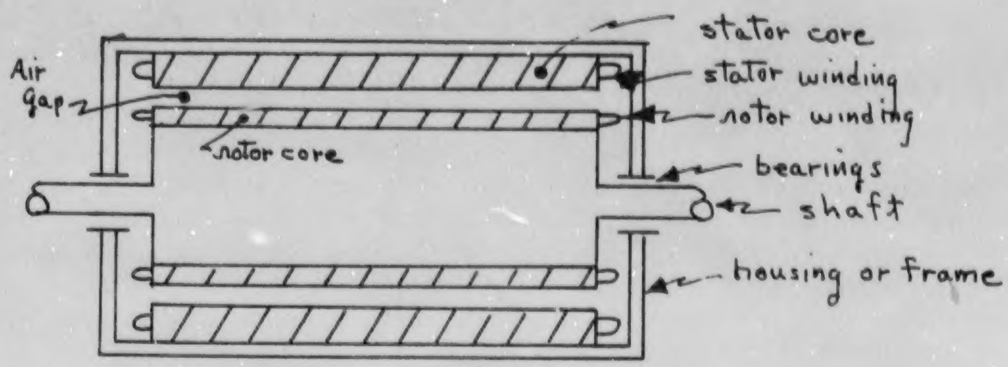


Figure III-1. The Basic Electric Machine

There are two magnetic cores - one common with the shaft and the other fixed to the frame, or housing. Many variations of this basic configuration are possible. The most common is to have the outer member stationary (the "stator") and the inner member, with shaft, the rotating member (the "rotor"). The two cores are separated by an air gap. There are unusual configurations involving stationary inner members and rotating frames, or, even, both members rotating with respect to a fixed reference and with respect to each other.

Each core has (usually) a winding. A machine with a permanent magnet field would be an exception. The winding consists of conductors carrying current and running parallel to the axis of the machine in the 'active' portion of the machine, i.e., the part that contains the narrow separation of rotor and stator called the air gap. The conductors are connected into windings. The exact nature of the connection and the type of winding is a function of the type and form of the electrical supply. In order to produce a net conversion of energy from electrical to mechanical form, the magnetic fields associated with the rotor and stator must be stationary with respect to each other even though they may not necessarily be stationary with respect to a stationary reference. The exact nature of the winding configuration and connection is of extreme importance to the designer. To the analyst, the connections are of importance only to the extent that the machine has specified input-output relationships and that the machine parameters are thus determined.

The machine parameters will characterize machine behavior under specific conditions existing within an overall system.

The three common types of windings are:

1. Coil winding
2. Polyphase connection of coils
3. Commutator winding

Coil windings are concentrated and the mmf and magnetic field associated with this type of winding acts along the axis of the coil winding. The winding itself consists of coils on all poles connected into a single circuit.

A polyphase winding consists of individual conductors distributed in slots around the periphery of the stator or rotor. The individual conductors actually are connected to form phase windings which may be coil windings and which, individually, have magnetic fields stationary in space (acting along the effective axis of the phase winding). However, polyphase windings are energized from alternating current sources so the magnitude of the magnetic field pulsates with a time variation even though it is stationary in space. Thus, the individual phase winding can be idealized by a single coil. The phase windings, if properly spaced and energized from alternating current of the proper phase, will cause a resultant magnetic field which (under steady state conditions) is of constant magnitude and revolving at constant angular velocity. For example, a three phase machine has three phase windings spaced $\frac{2}{3}\pi$ electrical radians apart and when energized from a 3 phase supply produces an mmf 3/2 greater than the maximum mmf in any one phase. The mmf rotates at the synchronous angular velocity of the supply frequency. This will be developed mathematically in a later section.

A commutator winding is composed of conductors located in slots and connected to commutator segments in some continuous sequence. The current flows into and out of this type of winding through carbon brushes (stationary)which bear on the copper commutator segments, or bars. The individual conductors comprising the winding are connected in some series-parallel combination to yield the desired current-voltage relationship, or rating, of the winding. There are always at least two and often times more, parallel paths through the winding. Commutator windings always are rotating windings. The purpose of the most simple commutator winding is to switch the revolving conductors, as they pass under the brushes, so that the conductors under any given pole always carry current in the same direction. The net result is that the axis of the commutator winding is always stationary with the axis of the pair of brushes.

Synchronous machines usually have a coil winding on the rotor (the "field") and a polyphase winding on the stator. Asynchronous, or "induction" machines have polyphase windings on both rotor and stator. (This statement is not true for single phase induction motors). Direct current machines have a coil winding ("field") on the stator and a commutator winding on the rotor, or as it is usually called, the armature.

In generalized machine analysis, we will find it helpful to define two main axis. The "direct" and the "quadrature" axis. In the case of the d.c. machine and of the synchronous machine, the direct axis is the axis along which the field (or excitation) mmf acts. The quadrature axis is $\pi/2$ radians advanced (in the direction of rotation) from the direct axis . These axis are usually referred to as the d and q axis.

Any machine, whether comprised of polyphase, coil, or commutator windings can be shown to be equivalent to an idealized machine consisting of coils located on the d and q axis. Even though the rotor rotates, the coils are fixed along these axis! To achieve this equivalence, it is necessary to convert, by suitable transformation, polyphase windings into equivalent rotating two phase windings. This transformation will be derived later. Since actual windings on the rotor have rotational voltages induced in them, we must ascribe special properties to the rotating winding. These two properties are possessed by a commutator winding. They are:

1. a current in the winding produces an mmf and a magnetic field which is stationary in space.

2. even though the mmf and the field are stationary in space, rotational voltages are induced in the coil.

The idealized machine we are developing will be referred to as a "Generalized Machine". The "Generalized Machine" will have only two poles and any results obtained will have to be altered to correspond to the actual machine.

At this point, the obvious advantages to the student of a "Generalized Machine" model should be apparent. If the various common machines can be represented by one model, then we need develop equations and analysis techniques only for the one model. Of course, some degree of skill and machine familiarity is necessary to relate model to actual machine types and vice-versa.

We are interested in dynamic, or transient, behavior of the machine with static, or steady state, situations as special cases of the dynamic situations. Dynamic behavior implies variation of performance with respect to time. Thus, our equations will be differential equations.

In order to bring to bear the sophisticated techniques of linear systems analysis to the behavior of our generalized machine, it will be necessary to ascribe certain other attributes which are not possessed by a physical machine. The attributes are really fundamental assumptions made for ease of solution of the differential equations which describe our model. They are:

1. No saturation exists in the magnetic circuit, i.e., the 'system' is linear and the principle of superposition can be applied. This enables us to sum mmfs, torques, voltages, etc. because by definition superposition means that the net effect is the sum of the causes.

2. Flux density harmonics are not present and the flux density is sinusoidally distributed in space. The main flux is considered to be determined by the fundamental component of the flux density. The radial line where the flux density is a maximum is the axis of the flux.

3. In calculating flux density from magnetomotove force, the mmf is considered to be acting only in the air gap. In other words, the iron is infinitely permeable and no mmf is required to sustain the flux in the iron.

4. The flux linking any winding is considered to consist of mutual and leakage flux. The leakage flux links only that particular winding. The mutual flux links all other windings lying on the same axis.

The above assumptions do introduce inaccuracies into the results obtained by mathematical analysis. Certainly, with nonlinear devices (which is always the case in an electromagnetic device comprised of iron) nothing yields more reliable results than actual test on the device itself. Such tests may not be possible under the desired conditions or in the system design stage. However, without the assumptions, analysis would be very nearly impossible even with computers because expressing the non-linearities mathematically introduces approximations. Previous work and experience indicates that the pattern of behavior emerges in spite of the assumptions! Correspondence between linear analysis results and test results is surprisingly close, as you can verify in laboratory work. Since some degree of saturation of the iron in the magnetic circuit is usually present, it is possible to refine results obtained by analysis if one makes a proper choice of variable to account for saturation effects. For closely coupled coils on an iron circuit, the assumption regarding a commun mutual flux is reasonably valid if different values of leakage are assigned for each winding or coil. The question of the effect of neglecting flux density space harmonics can be dealt with only if the type of machine is considered. For example, in d.c. machines, it is total flux per pole which is of concern in the steady state torque and voltage constants. However, under transient conditions that is no longer ture. There are other effects (to be dealt with later) which tend to "swamp out" the errors introduced by the assumption of sinusoidal flux density. In a.c. machines, the harmonics of the flux density are rotating at a speed other than the speed of the rotor flux density and thus are not effective in producing average torque. The value of instantaneous torque does vary, however, and some parasitic effects manifest themselves (noise, etc.). Also, of course, the flux density harmonics result in harmonics in the induced (both transformer and rotational) voltages.

To summarize the effect of the assumptions, one can only agree that they do result in errors analytical results. It is not possible to assign magnitude of errors without comparison with actual test results. The analytical results do give reasonably close results and yield valuable information to the system designer. Without the assumptions, analysis would be very nearly impossible.

We can represent, diagramatically, a generalized machine model as shown in Figure III-2 . This particular model has only 4 coils - 2 in each axis with one in each axis on the rotor and one in each axis on the stator. We will use only a 4 coil machine in our derivations in order to reduce the number of equations involved in the analysis. Some machines, such as the most simple d.c. machine will have only an F and a Q coil. A "cross field" type of d.c. machines may very well have additional F coils (usually designated as F_1, F_2,...,etc.) A synchronous machine often times will have additional circuitry on the rotor (damper circuits) and these are usually denoted as KQ and KD windings. However, a thorough understanding of the basic 4 coil model is essential to analysis of more complex configurations. The 4 coil model is a compromise between the completely general case and a simplified configuration without obscuring complexity.

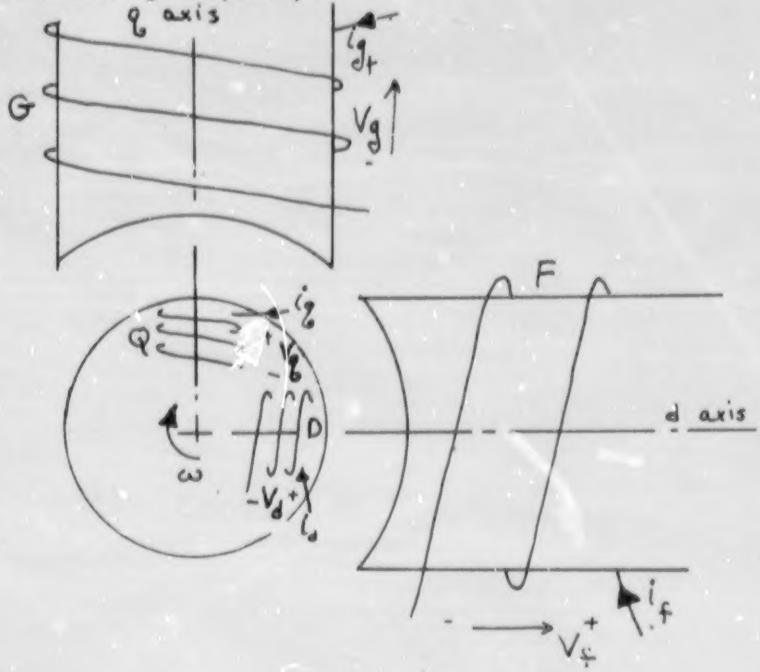


Figure III-2. 4 Coil Generalized Machine Diagram

In our derivation, we will assume the machine is rotating clockwise (cw) and that it is operating as a motor. Thus, positive current is the current that flows as a result of an external source and positive torque is torque in the direction of rotation. Further, we will resort to the per unit system, or basis, for defining the various parameters. This is discussed in the next section.

III.2 **THE PER UNIT SYSTEM:** The use of ratios to express results as normalized dimensionless quantities is quite common. Such ratios as percent efficiency, percent regulation, percent of rated load, power factor, etc. are widely used. If a quantity such as voltage drop, is expressed as per cent (or per unit) of rated voltage it is much more meaningful than if it is expressed in terms of its actual value. This is especially true if one is attempting to compare the performance, or the parameters, of a system with that of another system of different rating. In addition, as a result of standardization trends in design of equipment, many of the performance characteristics and parameters of machines are almost constant over a wide range in ratings if they are expressed as ratios. The use of such quantities has been applied to circuit analysis - greatly simplifying calculations in circuits involving transformers and circuits coupled magnetically. We can summarize the advantages as follows:

1. The use of per unit values facilities scaling and programming computers used for system studies.
2. The use of per unit values in problems solution yields results that are generalized and broadly applicable.
3. The solution of networks containing magnetically coupled circuits is facilitated. For example, with the proper choice of unit, or base, quantities the mutual inductance in per unit is the same regardless of which winding it is viewed from and regardless of the turns ratio of the windings.
4. Since the constants of machines, transformers and other equipment lie within a relatively narrow range when expressed as a fraction of the equipment rating one can make "educated" guesses as to probable value of per unit constants in the absence of definite design information. This is of assistance to the analyst when operating without complete device information.

It seems preferable to use per unit rather than per cent because of the problem encountered when two per cent quantities are multiplied together. For example, if:

$$A = 0.05 \text{ per unit or } 5\%$$

$$B = 0.20 \text{ per unit or } 20\%$$

and we take the product, $(A)(B) = (0.05)(0.20) = 0.01$ per unit. If we use percentages, $(A)(B) = (5)(20) = 100\%$, whereas it should be 1%. We will use per unit in this course and we will deal with quantities such as current I, voltage V, power P, reactive power Q, voltamperes VA, resistance R, reactance X, impedance Z, etc.

By definition,

$$\text{Quantity in per unit} = \frac{\text{actual value of the quantity}}{\text{base value of the quantity}} \qquad \text{(III-1)}$$

Actual quantity refers to its value in volts, ohms or whatever is applicable. Within limits, the base values may be chosen as any convenient number. However, for machine analysis they are usually chosen based on the rating of the machine. In power systems analysis, there are many factors that enter into the choice. We will not attempt a discussion of these factors at this time because our primary concern here is the analysis of machines.

We must select the base values so that the fundamental laws of electrical phenomenon are still valid in the per unit system. Ohm's law states that: (in actual values)

$$V = ZI \qquad \text{(III-2)}$$

If we select any two values for V_b, I_b and Z_b (where the subscript b denotes base values) then the third value is determined by the relationship

$$V_b = Z_b I_b \qquad \text{(III-3)}$$

dividing (III-2) by (III-3) yields

$$\frac{V}{V_b} = \left(\frac{Z}{Z_b}\right)\left(\frac{I}{I_b}\right) \qquad \text{(III-4)}$$

which is, according to (III-1)

$$V_{pu} = Z_{pu} I_{pu} \qquad \text{(III-5)}$$

where the subscript pu denotes per unit value.

It is common practice to select the voltampere base, VA_b, and the voltage base, V_b. Then, for single phase circuits

$$I_b = \frac{VA_b}{V_b} \quad \text{and} \quad Z_b = \frac{V_b}{I_b} \qquad \text{(III-6)}$$

$$Z_b = 2\pi f_b L_b = \omega_b L_b \qquad \text{(III-7)}$$

where L_b is base inductance, f_b is base frequency and ω_b is base electrical angular velocity.

It should be noted that 1.0 per unit power corresponds to the voltampere rating of the machine - not the actual power rating. These values are the same in a d.c. machine but may be different in an a.c. machine. As an example, if we had an a.c. machine rated 10000 kva, 0.85 p.f., the rated power is 8500 kw. However, 1.0 per unit power is taken as 10000. Thus, on a per unit basis, rated power is 0.85, not 1.0 per unit.

We will now examine flux and flux linkages and how they relate to the per unit system.

Consider two coils coupled as shown in Figure III-3. Each coil is assumed to be concentrated. By concentrated, we mean that the same flux links all turns in either coil. However, since the coils themselves exist as separate entities, not all of the flux which links one coil also links the other coil. We can define the flux linking coils as the main, or mutual flux, ϕ_m and the flux which links only a particular coil as the leakage flux of that coil, $\phi_{1\ell}$, $\phi_{2\ell}$.

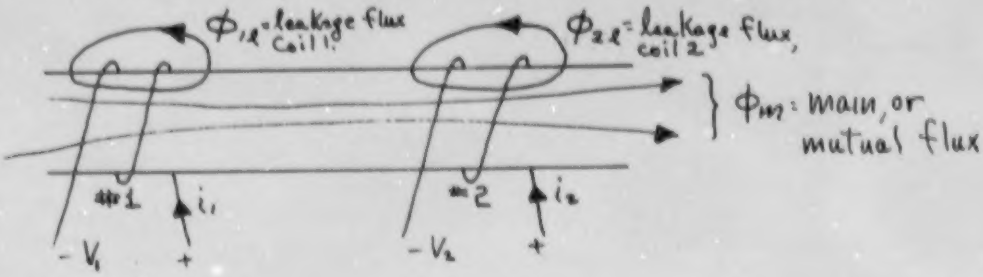

$\phi_{1\ell}$ = leakage flux Coil 1
$\phi_{2\ell}$ = leakage flux, coil 2
ϕ_m: main, or mutual flux
#1
i_1
#2
i_2
- V_1
+
- V_2
+

Figure III-3. Mutually Coupled Coils

Thus, there are three fluxes present in Figure (III-3).

ϕ_m = mutual, or main flux - linking both coils, resulting from both i_1 and i_2

$\phi_{1\ell}$ = leakage flux, due to current i_1 and linking coil 1 but not linking coil 2

$\phi_{2\ell}$ = leakage flux, due to current i_2 and linking coil 2 but not linking coil 1.

In the definitions used here, V represents voltage impressed on the coil from an external source and the current i is the current that flows as a result of V. Thus, if V and i are positive the device absorbs electric power.

The value of flux linkages, λ_{m1}, in coil 1 due to current i_2 in coil 2 is:

$$\lambda_{m1} = L_{12} i_2 \tag{III-9}$$

where L_{12} is the mutual inductance of the two coils.

Similarly, the value of flux linkages, λ_{m2}, in coil 2 due to current i_1 in coil 1 is:

$$\lambda_{m2} = L_{21}\, i_1 \tag{III-10}$$

Converting (III-9) and (III-10) to per unit values yields:

$$\lambda_{m1\ pu} = \frac{\lambda_{m1}}{\lambda_{1b}} = \frac{L_{12}\, i_2}{L_{1b}\, i_{1b}} = \frac{L_{12}\, i_{2b}\, i_{2\ pu}}{L_{1b}\, i_{1b}} \tag{III-11}$$

and:

$$\lambda_{m2\ pu} = \frac{\lambda_{m2}}{\lambda_{2b}} = \frac{L_{21}\, i_1}{L_{2b}\, i_{2b}} = \frac{L_{21}\, i_{1b}\, L_{1\ pu}}{L_{2b}\, i_{2b}} \tag{III-12}$$

rearranging (III-11) and (III-12) to obtain the ratio of per unit flux linkages in 1 coil to per unit current in the other coil yields:

$$\left(\frac{\lambda_{m1}}{i_2}\right)_{pu} = \frac{L_{12}\, i_{2b}}{L_{1b}\, i_{1b}} \quad ; \quad \left(\frac{\lambda_{m2}}{i_1}\right)_{pu} = \frac{L_{21}\, i_{1b}}{L_{2b}\, i_{2b}} \tag{III-13}$$

It is desirable to have the per unit flux linkages per unit current the same in either coil. (In other words, the per unit mutual inductance is the same when viewed from either coil!) We can ascertain the conditions necessary to achieve this by equating the ratios in (III-13). Also we recognize that in actual values, $L_{12} = L_{21}$. Further, we will divide each side of the equations by ω_b. Thus:

$$\frac{i_{2b}}{\omega_b\, L_{1b}\, i_{1b}} = \frac{i_{1b}}{\omega_b\, L_{2b}\, i_{2b}} \tag{III-14}$$

Recognizing that:

$$\omega_b L_b = Z_b \tag{III-15}$$

and:

$$Z_b \, i_b = V_b \tag{III-16}$$

we can rearrange (III-14) to yield:

$$V_{b2} \, i_{b2} = V_{b1} \, i_{b1} \tag{III-17}$$

In words, the result of this is that if we choose the same voltampere base for each coil, equal values of per unit current in the coils result in equal values of mutual flux linkages and the per unit value of mutual inductance is the same when viewed from either coil.

One other quantity requires special consideration. That is the base value of electrical angular velocity. This must be chosen as equal to 1.0 if the time relationships are to be preserved. That this is so can be seen from the following example.

Example

An R-L circuit is energized from a source, v = V sin ωt as shown in Figure III-4. The solution for current, i(t) is:

$$i(t) = \frac{V}{R} \left\{ \frac{\frac{\omega L}{R}}{1 + \left(\frac{\omega L}{R}\right)^2} \, \epsilon^{-\frac{t}{L/R}} + \frac{\sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2}} \right\} \tag{III-18}$$

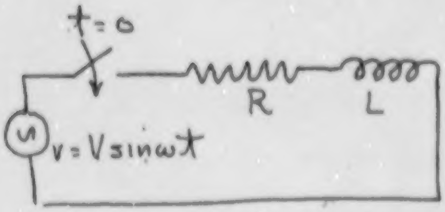


Figure III-4. R-L Circuit Energized from Sinusoidal Source

We can divide each side of the equation by I_b and can divide numerator and denominator of various ratios by Z_b as shown in (III-20). Recall that:

$$Z_b = R_b \quad \text{and} \quad Z_b = \omega_b L_b \qquad \text{(III-19)}$$

$$\frac{i(t)}{I_b} = \frac{V}{R\, I_b} \left\{ \frac{\dfrac{\omega L/\omega_b L_b}{R/R_b}}{1 + \left(\dfrac{\omega L/\omega_b L_b}{R/R_b}\right)^2}\, \epsilon^{-\frac{\frac{t}{L/\omega_b L_b}}{R/R_b}} + \frac{\sin\left(\omega/\omega_b t - \tan^{-1} \dfrac{\omega L/\omega_b L_b}{R/R_b}\right)}{\sqrt{1 + \left(\dfrac{\omega L/\omega_b L_b}{R/R_b}\right)^2}} \right\} \qquad \text{(III-20)}$$

Note that:

$$\frac{i(t)}{I_b} = i(t)\ \text{pu} \qquad \text{(III-21)}$$

$$\frac{V}{R\, I_b} = \frac{V}{R\left(\frac{V_b}{R_b}\right)} = \frac{V_{pu}}{R_{pu}} \qquad \text{(III-22)}$$

$$\frac{\frac{\omega L}{\omega_b L_b}}{\frac{R}{R_b}} = \frac{\omega_{pu} L_{pu}}{R_{pu}} = \left(\frac{\omega L}{R}\right) \text{actual} \qquad \text{(III-23)}$$

$$\frac{\frac{L}{\omega_b L_b}}{R/R_b} = \frac{L_{pu}}{\omega_b R_{pu}} = \frac{L_{pu}}{R_{pu}} = \left(\frac{L}{R}\right)\text{actual } \underline{\text{if}}\ \omega_b = 1.0 \qquad \text{(III-24)}$$

From (III-21) through (III-24) we can write (III-20) as:

$$i(t)\ \text{pu} = \frac{V_{pu}}{R_{pu}} \left\{ \frac{\dfrac{\omega_{pu} L_{pu}}{R_{pu}}}{1 + \left(\dfrac{\omega_{pu} L_{pu}}{R_{pu}}\right)^2}\, \epsilon^{-\frac{R_{pu} t}{L_{pu}}} + \frac{\sin\left(\omega t - \tan^{-1} \dfrac{\omega_{pu} L_{pu}}{R_{pu}}\right)}{\sqrt{1 + \left(\dfrac{\omega_{pu} L_{pu}}{R_{pu}}\right)^2}} \right\} \qquad \text{(III-25)}$$

This presupposes that $\omega_b = 1.0$. Let us apply some actual values and check results. For example, if:

$$V = 100, \; R = 50 \text{ ohms}, \; L = 10 \text{ henrys}, \; \omega = 5 \text{ rad/sec}$$

From (III-18):

$$i(t) = \varepsilon^{-t/0.2} + \sqrt{2}\, \sin(5t - 45^{\circ}) \qquad \text{(III-26)}$$

In per unit, we will arbitrarily choose $VA_{base} = 400$, $V_b = 200$, $\omega_b = 1.0$. Then;

$$I_b = \frac{VA_{base}}{V_b} = \frac{400}{200} = 2.0 \text{ amperes}$$

$$R_b = Z_b = L_b = \frac{200}{2.0} = 100 \quad (\text{if } \omega_b = 1.0)$$

and the circuit values, in per unit become:

$$R_{pu} = \frac{50}{100} = 0.5 \qquad V_{pu} = \frac{100}{200} = 0.5$$

$$L_{pu} = \frac{10}{100} = 0.1 \qquad \omega_{pu} = \frac{5}{1.0} = 5$$

Substituting values in (III-25);

$$i(t)\text{ pu} = \frac{0.5}{0.5} \cdot \left\{ \frac{\frac{5 \times 0.1}{0.5}}{1 + (\frac{5 \times 0.1}{5})^2} \varepsilon^{-\frac{0.5}{0.1}t} + \frac{\sin(5t - \tan^{-1}\frac{5 \times 0.1}{.5})}{\sqrt{1 + (\frac{5 \times 0.1}{0.5})^2}} \right\} \qquad \text{(III-27)}$$

$$= \frac{1}{2}\varepsilon^{-t/.2} + \frac{\sqrt{2}}{2}\sin(5t - 45^{\circ}) \qquad \text{(III-28)}$$

Now,

$$I_{act} = (I_{pu})\, I_{base}$$

$$= (\frac{1}{2})(2.0) = 1.0 \text{ ampere}$$

and

$$= \frac{\sqrt{2}}{2}(2.0) = \sqrt{2} \text{ amperes}$$

which is the same as the results in (III-26). Note that the time constant and angular velocity are the same in (III-26) and (III-28).

We can write equations describing the phenomena in coupled circuits, using a per unit system of values if we choose the voltampere base the same for each coil and we choose base angular velocity equal to 1.0.

For the mutually coupled coils of Figure III-3, let:

r_1, r_2 = per unit resistance of coils 1 and 2
L_{12} = per unit mutual inductance
ℓ_1, ℓ_2 = per unit leakage inductance

Applying Kirchoff's voltage summation law for each coil yields:

$$e_1 = r_1 i_1 + (L_{12} + \ell_1)\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \qquad \text{(III-29)}$$

$$e_2 = r_2 i_2 + (L_{12} + \ell_2)\frac{di_2}{dt} + L_{12}\frac{di_1}{dt} \qquad \text{(III-30)}$$

We can define coil self inductance in terms of the mutual and leakage inductances as follows:

$$L_{11} = L_{12} + \ell_1 \qquad \text{(III-31)}$$

$$L_{22} = L_{22} + \ell_2 \qquad \text{(III-32)}$$

We can rewirte (III-29, -30) as, in terms of self and mutual inductances:

$$e_1 = r_1 i_1 + L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \qquad \text{(III-33)}$$

$$e_2 = r_2 i_2 + L_{12}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} \qquad \text{(III-34)}$$

We will make use of these relationships many times in the analysis of electric machines.

Since we will be concerned with the electromechanical dynamics of the various machines we must also define speed, torque, and inertia in terms of the per unit system.

The per unit value of speed is the actual value, in rad/sec because base was taken as 1.0 in order to preserve the proper time relationship. Actual speed may be denoted by ω and may be above or below rated. The per unit value of torque is that torque which produces unit power at nominal speed.

The moment of inertia in the per unit system is based on the usage of a defined quantity, H, called the inertia constant. The inertia constant turns out to be a very useful number because its value varies over a relatively small range for a wide range of machine designs of different sizes and speeds. For example, synchronous machines have an inertia constant which lies (usually) between two and six. In the absence of specific data on the moment of inertia for a given machine, one can closely estimate the value of H based on the machine type and rating.

We will now investigate the inertia constant, H, and its relationship to actual moment of inertia. In the derivation that follows, these symbols are used:

δ = electrical angle, in radians, from some reference

δ_m = mechanical angle, in radians, from the same reference

J = polar moment of inertia, newton-meter-sec^2

ω_o = mechanical speed, radians/sec

ω_i = electrical speed, radians/sec

f = electrical frequency, cycles/sec

p = number of poles on machine

H = number of poles on machine

Now:

$$f = \frac{\omega_i}{2\pi} \qquad \text{(III-35)}$$

and:

$$\omega_i = \frac{p}{2}\omega_o \qquad \text{(III-36)}$$

Therefore:

$$f = \left(\frac{\omega_o}{2\pi}\right)\left(\frac{p}{2}\right) \qquad \text{(III-37)}$$

and:

$$\delta = \frac{p}{2}\delta_m \tag{III-38}$$

By definition:

$$H = \frac{\text{energy stored in rotor}}{\text{rated volt-amperes of the machine}} = \frac{1}{2}\,\frac{J\omega_o^2}{VA_{rated}} \tag{III-39}$$

For situations where unit power is chosen different from VA_{rated}, we define that power as VA_{base}.

Now $J\omega_o \frac{d^2\delta_m}{dt^2}$ is a power term, in watts, corresponding to inertia power in an electromechanical system. To put this power on a per unit basis, divide by VA_{base}. Thus, per unit inertia power is:

$$\frac{J\omega_o}{VA_{base}}\,\frac{d^2\delta_m}{dt^2} \tag{III-40}$$

Solving equation (III-39) for $J\omega_o$, we have:

$$J\omega_o = \frac{2(VA_{rated})\,H}{\omega_o} \tag{III-41}$$

If this is substituted in (III-40) we have the per unit inertia in terms of mechanical angle or speed.

$$\frac{2(VA_{rated})\,H}{\omega_o\,(VA_{base})}\,\frac{d^2\delta_m}{dt^2} = \frac{2H}{\omega_o}\left(\frac{VA_{rated}}{VA_{base}}\right)\frac{d\omega_o}{dt} \tag{III-42}$$

Many problems are expressed in terms of electrical angle, δ, and electrical frequency, f. From (III-37, -38, -42)

$$2H\left(\frac{VA_{rated}}{VA_{base}}\right)\frac{1}{\frac{2\pi f}{p/2}}\,\frac{1}{p/2}\,\frac{d^2\delta}{dt^2} = \left(\frac{VA_{rated}}{VA_{base}}\right)\frac{H}{\pi f}\,\frac{d^2\delta}{dt^2} \tag{III-43}$$

Another often used defined quantity is called the "Per Unit Moment of Inertia", M. Thus, by analogy,

$$M = \left(\frac{VA_{rated}}{VA_{base}}\right) \frac{H}{\pi f} \qquad \text{(III-44)}$$

This M is based on electrical angle in radians. M is sometimes defined in terms of electrical degrees in which case π is replaced by 180 in equation (III-44).

The moment of inertia of machines is often expressed as Wk^2 (weight times the square of the radius of gyration) in English units of lb.-ft.^{2}. To relate WK^2 to H, we can use the formula,

$$H = \frac{(Wk^2)(0.231)(rpm)^2}{(kVA_{base})\ 10^6} \qquad \text{(III-45)}$$

With the concept expressed here we can express the per unit power, corresponding to inertia torque, in an electromechanical system as

$$M \frac{d^2\delta}{dt^2} \text{ per unit power} \qquad \text{(III-46)}$$

where the angle δ is in electrical radians, and M is defined in (III-44).

In order to mathematically describe our defined model (the generalized machine) we must determine magnitude and polarity of induced voltages - both rotational and transformer types - and of the torques on the rotor. This will be done in the subsequent sections.

III.3 INDUCED VOLTAGES: In our model, all coils lie on either the d (direct) or q (quadrature) axis. Since the machine is assumed to be linear, we can consider the phenomena of direct axis and quadrature axis flux acting on the two possible coil configurations in one axis, i.e., either rotating or stationary. The net effect is the sum of the individual effects obtained by the principle of superposition. The results obtained can be applied to coils in the other axis by analogy.

It was previously specified that positive currents and voltages correspond to motor action, i.e., the machine absorbs power from the electrical sources. In order to determine actual polarities and directions, we must indicate the direction of current flow in the various coils, shown in Figure III-2. The defined directions are as shown in Figure (III-5).

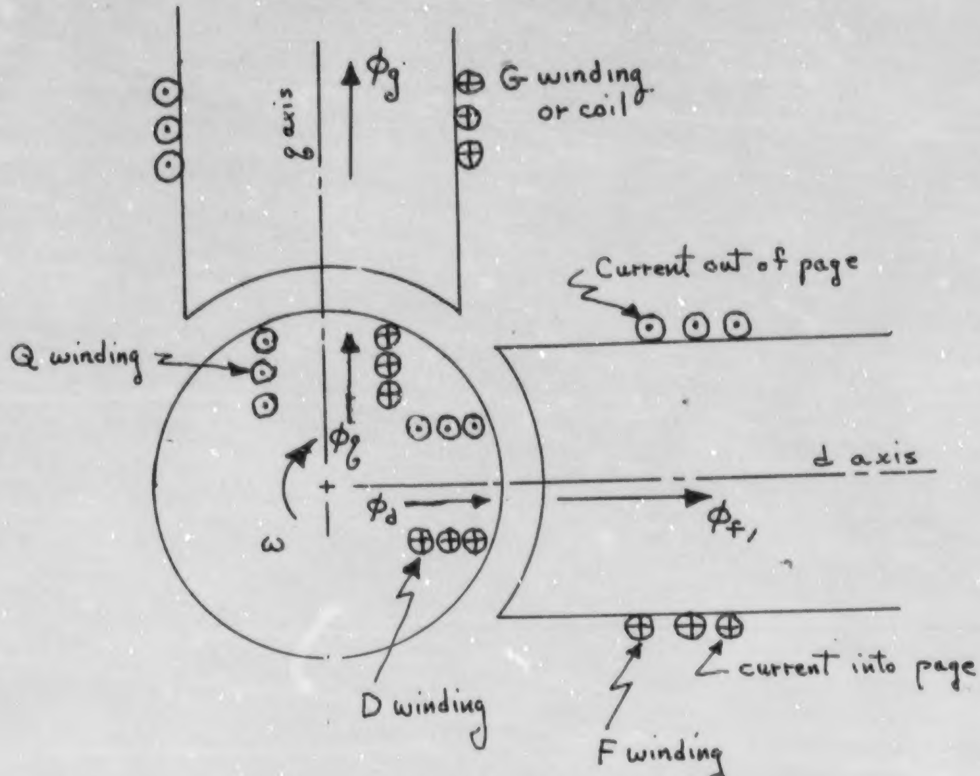

ϕ_g
g axis
G winding or coil
Current out of page
Q winding
ϕ_q
d axis
ϕ_d
ϕ_{f_1}
ω
D winding
F winding
current into page

Figure III-5. Defined Direction of Positive Current Flow and Rotational Direction for the Generalized Machine (Motor)

The various inductances in the machine in Figure III-5 will be denoted as follows

Mutual Inductances

L_{df} , L_{gq}

Leakage Inductances

ℓ_f, ℓ_d; ℓ_g, ℓ_q

Self Inductances

L_f, L_d ; L_g, L_q

The relationship between these various inductances is:

$$L_q = L_{qg} + \ell_q \tag{III-47}$$

$$L_g = L_{qg} + \ell_g \tag{III-48}$$

$$L_d = L_{df} + \ell_d \tag{III-49}$$

$$L_f = L_{df} + \ell_f \tag{III-50}$$

Current in the various coils causes flux linkages in every coil along that axis. We will denote flux linkages as follows;

Total Coil Flux Linkages

$\lambda_q , \lambda_g , \lambda_d . \lambda_f$

The mutual flux linkages correspond to the flux in the air gap and which is common to all coils on that axis. The total coil flux linkages correspond to the flux linking a specific coil only and is the sum of the mutual and leakage components.

Flux linkage is the product of current and inductance. We can express the various flux linkages as:

$$\lambda_q = L_q i_q + L_{gq} i_g \tag{III-51}$$

$$\lambda_g = L_g i_g + L_{gq} i_q \tag{III-52}$$

$$\text{q axis air gap} = L_{gq} (i_q + i_g) \tag{III-53}$$

$$\lambda_d = L_d i_d + L_{df} i_f \tag{III-54}$$

$$\lambda_f = L_f i_f + L_{df} i_d \tag{III-55}$$

$$\text{d axis air gap} = L_{df} (i_d + i_f) \tag{III-56}$$

A positive sign for the flux linkage in the air gap denotes flux directed outward in the convention for current polarity chosen.

The effect of d axis flux linkages on coil D will be analyzed for the situation where the flux is varying with respect to time and for the situation wherein the conductors are moving with respect to the flux.

The coil on the rotating member (the rotor) is assumed to be distributed over the periphery of the rotor. The coil is also assumed to be formed by a commutator type winding. Suffice to say, at this point, that the commutator winding has brushes which bear on copper commutator segments and that the net effect of the commutator is to perform a swtiching operation which renders the revolving coil stationary in space. As the winding rotates, the brushes switch direction of current flow and reconnects the winding continuously. Referring to Figure III-6, we can describe what happens to any specific conductor by stating that the conductor is connected with a certain polarity as it passes through $\theta = 0$. It maintains this polarity for π electrical radians at which time its relative polarity in the circuit is reversed.

As the rotor rotates, all conductors lying between 0 and π radians have current flowing in them in the same direction, i.e., into the page in Figure III-6. Also, any induced voltages, due to rotations will be induced in a direction so as to be additive. Conductors lying between π and 2π radians have current flowing in the opposite direction and the induced voltages are in such a direction as to be additive with the induced voltages in the other part of the winding. The commutator winding behaves as a stationary coil with fixed current direction but it also has the capability of having a rotational voltage induced in the coil.

Consider that the coil is distributed over the periphery with a density of z conductors per radian.

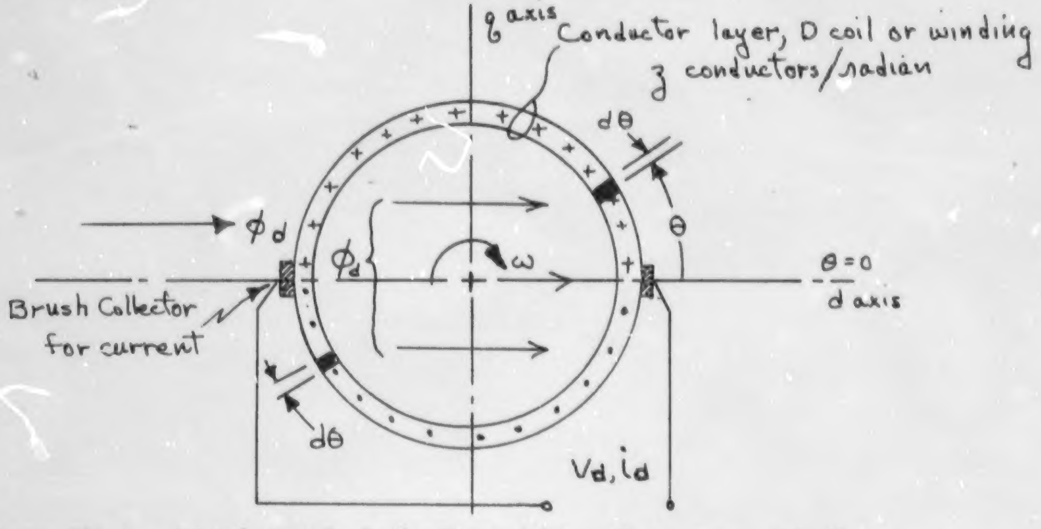


Figure III-6. Distributed D Coil, Formed by a Commutator Winding

The shaded area in the conductor layer represents some turns of a coil with coil sides at θ and at $(\theta + \pi)$ radians.

In practice, the flux density B associated with flux ϕ_d can have any distribution. However, regardless of space distribution it can be resolved into an infinite series of trigonometric functions by means of Fourier Analysis techniques. As stated previously, in this analysis we will consider only the linear case and effects related to the fundamental. Since the flux density is a maximum along the direct axis $(\theta = 0)$ we can express it as $B = B_m \cos\theta$. If the rotor length is ℓ and its radius r, we can relate flux and flux density as follows:

$$\phi_d = \int_{-\pi/2}^{\pi/2} B \ell r \, d\theta = \int_{-\pi/2}^{\pi/2} B_m \cos\theta \, d\theta = 2 B_m \ell r \qquad \text{(III-57)}$$

The flux linking the turn (shown shaded in Figure III-6) is:

$$\phi = \int_{+(\pi+\theta)}^{\theta} B_m \cos\theta \, \ell r \, d\theta = 2 B_m \ell r \sin\theta = \phi_d \sin\theta \qquad \text{(III-58)}$$

If the flux is varying with respect to time, a transformer voltage, e_t, is induced in the turn. It is:

$$e_t = \frac{d}{dt} (\phi_d \sin\theta) \qquad \text{(III-59)}$$

The shaded area contains $z d\theta$ conductors. The summation of all the conductors lying between $\theta = 0$ and $\theta = \pi$ will yield the total voltage induced in the winding (assuming the conductors are connected in a series additive configuration). Denote:

e_{dt} = sum of voltages induced in the series connected winding

Then:

$$e_{dt} = \int_0^{\pi} \frac{d}{dt} (\phi_d \sin\theta) z \, d\theta \qquad \text{(III-60)}$$

$$= \frac{d}{dt} \int_0^{\pi} \phi_d z \sin\theta \, d\theta = \frac{d}{dt} (2 \phi_d z) \qquad \text{(III-61)}$$

From Faraday's law, the voltage induced by time varying flux is simply equal to the time rate of change of flux linkages linking the coil. Thus,

$$e_{dt} = \frac{d\lambda_d}{dt} \qquad \text{(III-62)}$$

where λ_d is the direct axis flux linkage. Faraday's induced voltage relationship enables us to relate total flux (associated with sinusoidal distribution of flux density), conductor density, and flux linkages by comparing (III-61) and (III-62), yielding:

$$2\,\phi_d\,z = \lambda_d \qquad \text{(III-63)}$$

This important relationship will be used later in this section.

Note that it is necessary to integrate only from 0 to π in order to include all turns because the elemental turn has two sides. From (III-61) it can be seen that a "transformer" voltage exists if the flux is changing in the rotor winding (formed by brushes) located along the axis of the time varying flux.

Now, consider the effect of the quadrature axis flux, ϕ_q, on the rotor circuit formed by brushes in the direct axis. The development would be similar except the flux density is a maximum at $\theta = \pi/2$ and would be expressed as $B = B_m \sin\theta$.

The flux linking the turn in Figure (III-6) would be:

$$\phi = \int_{+(\pi+\theta)}^{\theta} B_m \sin\theta\,\ell r\,d\theta = -2\,B_m\,\ell\,r\cos\theta = -\,\phi_q\cos\theta \qquad \text{(III-64)}$$

The induced transformer voltage in the turn, e_t, is:

$$e_t = +\frac{d}{dt}\left(-\,\phi_q\cos\theta\right) \qquad \text{(III-65)}$$

and e_{Qt}, the sum of the transformer voltages in all the turns due to q axis flux, is:

$$e_{qt} = -\frac{d}{dt}\int_0^{\pi} \phi_q\,z\cos\theta\,d\theta = 0 \qquad \text{(III-66)}$$

No transformer voltage exists in the direct axis rotor winding due to time varying quadrature axis flux (and vice versa, of course). We could have forseen this by noting that the two windings are in quadrature and that the mutual inductance coupling coefficient is zero.

The polarity of the induced transformer voltage can be determined from the fact that it is always in the direction such that, if current could flow, it would flow in a direction such as to oppose a change in flux linkages. Thus it opposes the driving voltage. For a coil with no rotational voltages, such as F, application of Kirchoff's law for voltage rises yields:

$$V_f = r_f i_f + \frac{d\lambda_f}{dt} \qquad \text{(III-67)}$$

To investigate the rotational, or "speed" voltage induced in the D coil, recall that this type of voltage is determined by: (in vector form)

$$e_{rot} = (\bar{v} \times \bar{B})d\ell \qquad \text{(III-68)}$$

where $\bar{v}$ is linear velocity, $\bar{B}$ is flux density and $d\ell$ is an element of conductor length. The bar on top denotes a vector quantity - one which has magnitude and direction. From electro-magnetic field theory, flux density leaves an interface between two media of different permeability perpendicular to the interface. In a machine, the flux density is thus perpendicular to the surface of the iron at the air gap. For small air gap we can assume that the conductor element is perpendicular to the flux density because its linear velocity is tangential to the surface. Since the conductor length is mutually perpendicular to the velocity and the flux density, (III-68) reduces to:

$$e_{rot} = B\ell v = B\ell r\omega \qquad \text{(III-69)}$$

This is the voltage induced in each conductor of the elemental coil shown shaded in Figure III-6. Since each turn has two conductor sides, each turn would have two times the value given in (III-69). There are $zd\theta$ conductors in the shaded area. Thus, for total rotational voltage in the shaded area,

$$e_{rot} = (B\ell r\omega)(2)(zd\theta) \qquad \text{(III-70)}$$

Total rotational voltage in the winding is obtained by summing over the angle from $\theta = 0$ to $\theta = \pi$. Thus:

$$e_{rot} = \int_0^{\pi} B\ell r\omega 2z \, d\theta \qquad \text{(III-71)}$$

$$= \int_0^{\pi} B_m \ell r\omega 2z \cos\theta d\theta = 0 \qquad \text{(III-72)}$$

We conclude that no rotational voltage exists if the flux density axis and the winding axis coincide.

In order to evaluate the rotational voltage in the direct axis due to quadrature axis flux, we can use equatiuon (III-72) except that the flux density is a sine function rather than a cosine function. (This because the flux is centered on the quadrature axis and is thus a maximum at $\theta = \pi/2$). With this change, (III-72) yields:

$$e_d \text{ rot} = \int_0^\pi B_m \ell r\omega 2z \sin\theta d\theta = 2(2B_m \ell r)z\omega \qquad \text{(III-73)}$$

recall that (from analogy with (III-57)):

$$\phi_q = 2 B_m \ell r \qquad \text{(III-74)}$$

Therefore, we can express (III-73) as:

$$e_{d \text{ rot}} = 2z\phi_q \omega \qquad \text{(III-75)}$$

By analogy to (III-63), we can express the rotational voltage induced in the direct axis rotor circuit due to quadrature axis flux as:

$$e_{d \text{ rot}} = \lambda_q \omega \qquad \text{(III-76)}$$

where:

$$\lambda_q = 2z \phi_q \qquad \text{(III-77)}$$

Similarly, the rotational voltage in the q axis, due to d axis flux is:

$$e_{q \text{ rot}} = \lambda_d \omega \qquad \text{(III-78)}$$

The direction, or polarity, of these rotational voltages, relative to driving voltage must be ascertained before mathematical models of the machine can be formulated.

We can summarize our investigation thus far as follows: In a rotating rotor coil with properties previously ascribed to such a coil, both rotational and transformer voltages can be induced. The transformer voltages are induced by the time varying flux along the axis of the winding. The rotational voltages are induced in the winding whose axis is in quadrature with the flux under consideration. Thus:

$$e_{dt} = \frac{d\lambda_d}{dt} \qquad e_{d\,rot} = \omega\ \lambda_q \qquad \text{(III-79)}$$

$$e_{qt} = \frac{d\lambda_q}{dt} \qquad e_{q\,rot} = \omega\ \lambda_d \qquad \text{(III-80)}$$

where the subscripts t and rot denote transformer and rotational, or speed voltages and q and d denote coils on the quadrature and direct axis, respectively. The F and G coils are stationary and rotational voltages thus cannot exist in them. Rotational voltages can exist in the rotor coils if flux is present in the other axis.

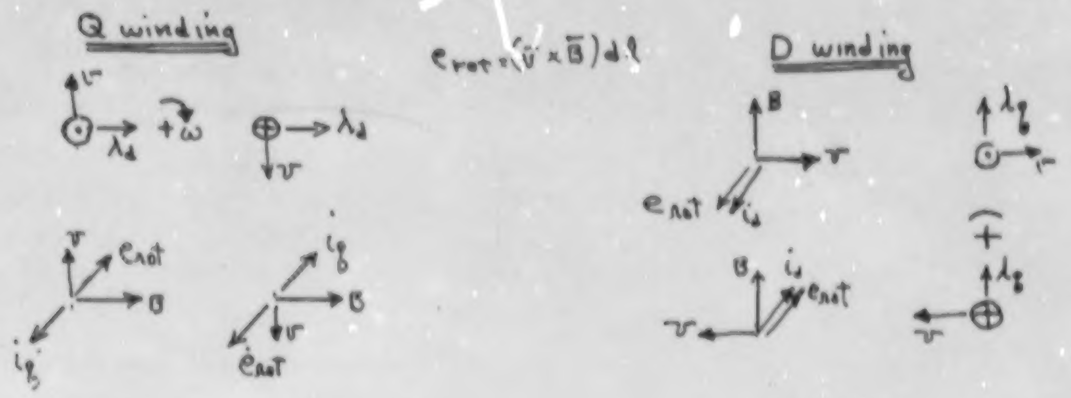
The equations for transformer voltages are valid under any consideration in so far as flux density space distribution is concerned. The equations for rotational voltage are valid for sinusoidal space distribution and the magnitude of the flux linkage, λ, is given by:

$$\lambda = 2\phi z \qquad \text{(III-81)}$$

where ϕ is total flux per pole, z is the number of conductors per electrical radian on the rotor periphery.

What if the flux density space distribution is not sinusoidal? As a matter of fact, most d.c. machines have a trapazoidal distribution at no load which becomes distorted as load increases. However, the spacial flux density distribution is periodic and can be resolved into a fundamental and harmonics of the fundamental. In Section II.1 on the elementary d.c. machine, it was shown that the constant relating rotational voltage and speed is actually proportional to the total flux per pole and is not a function of distribution.

To determine the direction of the induced rotational voltages, refer to equation (III-68). An application of the vector implications of this equation to windings D and Q is shown in Figure III-7. Only one turn of each of these coils is shown in order to simplify the diagram.


Q winding
$e_{rot} = (\bar{v} \times \bar{B}) \cdot dl$
D winding
v
λ_d
$+\omega$
λ_d
v
B
v
e_{rot}
i_d
λ_q
v
v
e_{rot}
B
i_q
i_q
B
v
e_{rot}
B
i_d
e_{rot}
v
$+$
λ_q
v

(a)
i_q and the induced rotational voltage are in opposite directions

(b)
i_d and the induced rotational voltage are in the same direction

Figure III-7. Rotational Voltage Polarities

From Figure III-7, the rotational voltage in the direct axis rotating winding acts in the same direction as i_d and thus has the same polarity as the driving voltage V_d. Similarly, the rotational voltage in the quadrature axis rotating winding is in opposition to i_q and has opposite polarity to V_q.

We can now write the voltage equations for the four coil machine as follows:

$$v_f = r_f i_f + \frac{d\lambda_f}{dt} \tag{III-82}$$

$$v_d = r_d i_d + \frac{d\lambda_d}{dt} + (-\omega\lambda_q) \tag{III-83}$$

$$v_q = r_q i_q + \frac{d\lambda_q}{dt} + \omega\lambda_d \tag{III-84}$$

$$v_g = r_g i_g + \frac{d\lambda_g}{dt} \tag{III-85}$$

where r_f, r_d, r_q, r_g are lumped resistances for the F, D, G, and Q windings.

III.4 DEVELOPED TORQUES: In order to evaluate torque at the shaft of the Generalized Machine, consider the total electrical power, P, supplied to the machine.

$$P = k \sum v\, i = k(v_f i_f + v_d i_d + v_q i_q + v_g i_g) \qquad \text{(III-86)}$$

The value of k and the reason for k will be discussed later. The individual winding power inputs are obtained by substituting the value of the various voltages from (III-82) through (III-85) into (III-86). Thus

$$v_d i_d = r_d i_d^2 + i_d \frac{d\lambda_d}{dt} - \omega i_d \lambda_q \qquad \text{(III-87)}$$

$$v_f i_f = r_f i_f^2 + i_f \frac{d\lambda_f}{dt} \qquad \text{(III-88)}$$

$$v_g i_g = r_g i_g^2 + i_g \frac{d\lambda_g}{dt} \qquad \text{(III-89)}$$

$$v_q i_q = r_q i_q^2 + i_q \frac{d\lambda_q}{dt} + \omega i_q \lambda_d \qquad \text{(III-90)}$$

The $i^2 r$ terms are, of course, ohmic power loss which appears as heat. Terms of the form:

$$i_d \frac{d\lambda_d}{dt} \qquad \text{(III-91)}$$

are terms expressing the time rate of change of stored field energy. To visualize that this is so, refer to the value of λ_d from (III-54) and substitute it in (III-91), yielding:

$$i_d \frac{d\lambda_d}{dt} = i_d L_d \frac{di_d}{dt} + i_d L_{df} \frac{di_f}{dt} \qquad \text{(III-92)}$$

Recall that stored magnetic energy is given by:

$$W_{fld} = \frac{1}{2} L\, i^2 \qquad \text{(III-93)}$$

and:

$$\frac{d}{dt} W_{fld} = L\, i \frac{di}{dt} \qquad \text{(III-94)}$$

Thus, terms representing time rate of change of stored field energy do not represent, or express, energy converted to mechanical form. Only the terms:

$$\omega i_d \lambda_q \quad \text{and} \quad \omega i_q \lambda_d \tag{III-95}$$

contribute to shaft power (after conversion to mechanical form) or are a result of shaft power before conversion to electrical form (as in a generator). The power converted to or from mechanical form is, from (III-86) through (III-95) then,

$$P = k\,(i_q \lambda_d - i_d \lambda_q)\,\omega \tag{III-96}$$

In the case of a motor this power is shaft output plus the mechanical losses of friction and windage. For a generator, this power is shaft mechanical power input minus the mechanical losses of friction and windage.

The electromechanical torque, in the mks system is:

$$T_e = \frac{P}{\omega} = k\,(i_q \lambda_d - i_d \lambda_q) \tag{III-97}$$

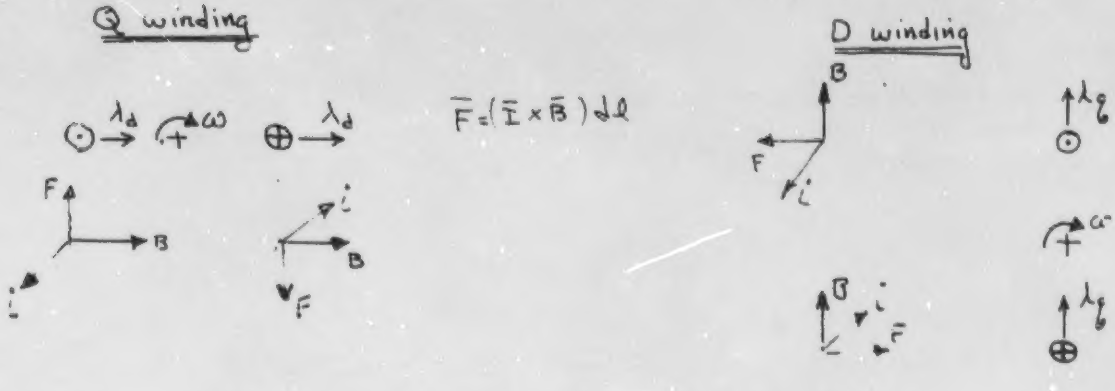
The value of k is chosen to make the per unit torque equal to 1.0 when unit torque and unit flux linkages are present. For the four coil machine, two main circuits are present, i.e., the D and Q coils. For this machine, k = 1/2. In general, k is the reciprocal of the number of main circuits.

To ascertain the direction of the electromechanical torque recall the basic relationship, from electromagnetic flux field theory, for force, $\overline{F}$, on an elemental length, $d\ell$, of a conductor lying in a magnetic field of intensity, $\overline{B}$, and carrying a current, $\overline{I}$. (The bar on top denotes a vector quantity, i.e., one which has both direction and magnitude).

$$\overline{F} = (\overline{I} \times \overline{B})\,d\ell$$

Torques are exerted on the rotor and stator (equal and opposite) but since the rotor is free to turn we shall consider the torques only on the rotor. Recall that the cross product of two quantities along the same axis is zero. Therefore, we need only consider the effect of λ_d on the Q winding and λ_q on the D winding in this machine. This was verified analytically in the previous section.

Figure III-8 depicts the force on selected inductors for the D and Q windings.



(a)
Torque resulting on winding is cw and is in the direction of the rotation

(b)
Torque resulting on winding is ccw and is opposite to the direction of rotation

Figure III-8. Torque Directions

The torque on the Q winding is in the direction of rotation. This is a motor; therefore, this is logically positive torque. The torque on the D winding is then negative torque because it is opposite to the direction of rotation. Equation (III-97) agrees algebraically with the physical reasoning presented here. This is the reason for defining the specific current directions in the D and Q windings.

We must also describe the electromechanical relationship between electromechanical torque, the moment of inertia (in per unit), and the torque required by the load. Load torque is usually a function of speed. From Newton's Law of dynamics for a mechanical system;

$$T_e = T_L(\omega) + M \frac{d\omega}{dt} \qquad \text{(III-98)}$$

where:

$T_L(\omega)$ = load torque

T_e = developed machine torque

M = moment of inertia of rotor and load

III.5 SUMMARY: We can now summarize the mathematical relations which model, or describe, the 4 winding generalized machine. Positive quantities denote a motor. The equations used are (III-51, -52, -54, -55) for the flux linkages, (III-82, -83, -84, -85) for the voltage-current relationships and (III-97, -98) for the electromechanical relationships:

$$V_f = r_f i_f + \frac{d\lambda_f}{dt} \qquad \text{(III-99)}$$

$$V_d = r_d i_d + \frac{d\lambda_d}{dt} - \omega\lambda_q \qquad \text{(III-100)}$$

$$V_q = r_q i_q + \frac{d\lambda_q}{dt} + \omega\lambda_d \qquad \text{(III-101)}$$

$$V_g = r_g i_g + \frac{d\lambda_g}{dt} \qquad \text{(III-102)}$$

$$T_e = k(i_q\lambda_d - i_d\lambda_q) \qquad \text{(III-103)}$$

$$T_e = T_L(\omega) + M\frac{d\omega}{dt} \qquad \text{(III-104)}$$

$$\lambda_f = L_f i_f + L_{df} i_d \qquad \text{(III-105)}$$

$$\lambda_d = L_d i_d + L_{df} i_f \qquad \text{(III-106)}$$

$$\lambda_q = L_q i_q + L_{qg} i_g \qquad \text{(III-107)}$$

$$\lambda_g = L_g i_g + L_{qg} i_q \qquad \text{(III-108)}$$

In subsequent chapters, we will relate specific types of machines to the generalized machine model developed here. Since we have derived the applicable equations, behavior can be analyzed by mathematical manipulations on the appropriate equations (III-99) through (III-108). We must study specific types of machines in order to determine which of these equations apply to a specific machine configuration. These equations are written for a motor. For a generator, the direction of current in the power windings, or coils would reverse. Also, of course, T_L would reverse (negative sign) because mechanical power flows into the shaft, rather than out as for the motor. T_e would have a negative sign because of the change in direction of the currents.

CHAPTER IV - D.C. MACHINES

IV.1 INTRODUCTION: The basic, essential, features of a d.c. machine are shown in Figure IV-1. Note that this is a two pole machine while, in practice, the machine may have multiple pairs of poles.

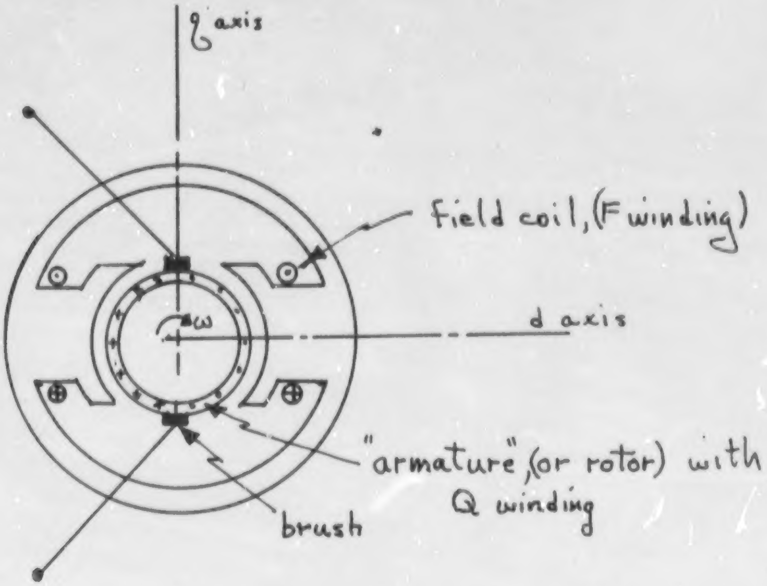
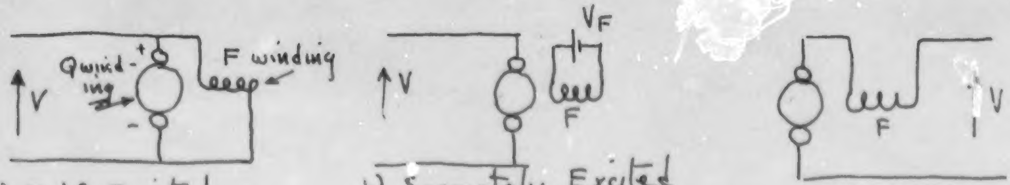


Figure IV-1. Schematic Representation of a Conventional D.C. Machine

Note that the rotor has been designated as the armature and the F winding designated as the "field". This is customary in d.c. machines. The conventional D.C. machine does not have a D winding on the rotor, nor does it have a G winding along the q axis on the stator. It would be difficult to imagine a more simple configuration for analysis.

The field winding (F winding) is either a concentrated coil consisting of a few turns of large cross sectional area wire or many turns of relatively fine, or small, cross sectional wire. The former is usually called a "series field" and the latter a "shunt field". Common types of electrical connection for the D.C. machine are shown in Figure IV-2.



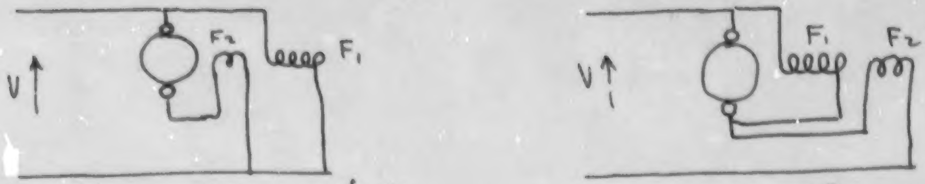
a) Self Excited Shunt Connection

b) Seperately Excited Shunt Connection

c) Series Connection

Figure IV-2. Common Types of Connections, D.C. Machines

The field winding provides "excitation", i.e., establishes the magnetic field, λ_d. Since flux linkages are proportional to the product of current and inductance and inductance is proportional to the square of the number of turns, the general idea in the shunt connection is to use a relatively large number of turns to keep the current (and power converted to heat) as low as possible. For the series wound machine, the current from the power circuit flows through the series winding. The current in the power circuit is on the order of 20-30 times larger than that which flows in the shunt field circuit in a well designed machine. From the standpoint of losses as heat, the coil for a series-wound machine should be designed with low ohmic resistance. There are other reasons why this is desirable (such as "regulation" - either voltage or speed) and these will be developed later. Since the current is relatively large, fewer turns can be made to yield the same value of flux linkages as a winding with many turns carrying a small value of current. The point of this discussion is that the field coil and the armature can either be in series or in parallel, but the field coil must be designed for the desired combination. Some d.c. machines have two field coils - one for series connection and one for shunt connection. These machines are classified as "Compound-wound" and would have two F windings, F_1 and F_2, as discussed earlier in connection with the description of the generalized machine. The shunt coil is usually much stronger than the series coil and the machine has characteristics similar to the shunt machine although modified by the load current through the series coil. The compound wound machine can be further classified as cumulative or differential compound depending on whether the series flux is in the same or in the opposite direction as shunt winding flux. The schematic representation for a compound wound machine is shown in Figure IV-3.



F₁ - shunt field
F₂ - Series field

a) "Long Shunt" Connection b) "Short Shunt" Connection

Figure IV-3. Compound Connections, D.C. Machines

In describing the shunt coil as being connected in parallel with the armature coil, we should perhaps describe it as being designed for connection if we so desire. That is to say, it is designed for rated voltage of the armature coil and may either be connected directly in parallel with the armature ("Self excited") or connected across a separate circuit of the same voltage rating as the rated voltage in which case it is "separately excited". These terms are especially appropriate when we refer to a generator or to a motor powered from a variable voltage source.

Examination of a large d.c. machine may indicate the presence of additional windings, such as series connected turns on small poles between the main poles. The main poles are located on the d axis. The small poles referred to here are located on the q axis. The purpose of these small poles, called interpoles, is to assist in the commutation process, i.e., the switching of conductor connection by the commutator as the rotor turns and commutator segments pass under the brushes. Another form of winding consists of inductors (or conductors) embedded in the main pole face and series connected. These are called "compensating" windings and their purpose is to compensate for the flux changes in the air gap that results from armature current and magnetic non linearity. This phenomena is referred to as "armature reaction". Both the commutating and the compensating windings act to correct phenomena which cause the machine to depart from an idealized machine and do not contribute to the energy conversion process. An understanding of their specific details is not essential to an analysis of machine behavior based on the linearized version of the machine.

The equations describing the generalized machine can be modified, or adapted, to the d.c. machine if we specify how the d.c. machine is connected and in what mode it operates, i.e., as a motor or generator. Several examples involving various configurations will be analyzed in this chapter. They are:

1. The Shunt Connected, Separately Excited Generator
2. The Shunt Connected, Separately Excited Motor.
3. The Series Connected Motor.
4. The Self Excited, Compound Connected DC Machine on Short Circuit.

One might reasonably ask why we choose a separately excited generator rather than self excited. There are two reasons - the first being that this is a common method of operation in control systems and the second being that the operation of a self excited machine acting as a generator depends on the nonlinearities present. With a linear machine (i.e., linear flux vs. excitation) the mathematics will convince you that it cannot work. The physical reasoning is that when load is switched on, the voltage across the field drops, excitation decreases, causing still more voltage drop, etc. until the voltage collapses completely. However, a real machine, with non linearity, does work. There are techniques available (graphical, by computer, and by iteration) which enable us to analyze self excited machines with non linearities.

IV.2 THE SHUNT CONNECTED, SEPARATELY EXCITED GENERATOR WITH SWITCHED LOAD:

In this analysis we will assume:

1. The generator is driven at constant rated speed, i.e., $\omega = 1.0$ per unit.
2. A load consisting of resistance, R_L and inductance, L_L is switched at $t = 0$.
3. We will excite the F coil from a constant voltage source and will collect D.C. Current from the Q coil. (The usual configuration).

Recall that the equations for the generalized machine (III-99) through (III-106), were written for motor action. For the generator, the current is in the opposite direction from that used for motor action. We will denote the terminal voltage and load current by the subscript a.

Thus:

$$e_q = e_a \qquad i_q = -i_a; \tag{IV-1}$$

Similarly,

$$L_q = L_a \qquad r_q = r_a; \tag{IV-2}$$

Note the absence of the D winding and the G winding. Also, since ω = constant, equation (III-105) becomes:

$$T_e = -T_L(\omega) \tag{IV-3}$$

The minus sign indicates mechanical power enters the shaft, rather than being taken from the shaft and the equality states that the mechanical power used to drive the machine is all converted to electrical form. This is true only if we neglect the mechanical losses in the machine (friction and windage losses). The appropriate equations from (III-99) through (III-108) can then be rewritten as follows:

$$v_f = r_f i_f + L_f \frac{di_f}{dt} \qquad \text{(IV-4)}$$

$$v_a + r_a i_a + L_a \frac{di_a}{dt} = \omega L_{df} i_f \qquad \text{(IV-5)}$$

$$\omega T_e = -\omega T_L(\omega) = P_{mech} \qquad \text{(IV-6)}$$

$$\omega(-i_a \lambda_d) = -\omega i_a i_f L_{df} = P_{mech} \qquad \text{(IV-7)}$$

where P_{mech} is the mechanical shaft power into the generator.

The F and Q windings are in quadrature. Therefore changes of current, i_a, in the Q winding do not affect the F winding since no magnetic coupling exists between them. At $t = 0$, i_f is in the steady state condition and has a value,

$$i_f = \frac{v_f}{r_f} \qquad \text{(IV-8)}$$

i_f remains constant at this value.

Taking the Laplace transform of (IV-5) yields:

$$v_a(s) + (r_a + L_a s)\, I_a(s) - L_a i_a(0) = \frac{\omega L_{df} i_f}{s} \qquad \text{(IV-9)}$$

From the problem statement, $i_a(0) = 0$. However, since the generator was rotating and excitation current i_f existed, a no load voltage existed at the terminals of the generator. From (IV-5) with $i_a = 0$,

$$v_a(0) = \omega L_{df} i_f \qquad \text{(IV-10)}$$

From (IV-10), i_f can be determined as a function of no load voltage. If this is done and substituted in (IV-9), we have:

$$v_a(s) + (r_a + L_a s)\, I_a(s) = \frac{v_a(0)}{s} \qquad \text{(IV-11)}$$

IN (VI-11) we have two variables, $v_a(s)$ and $i_a(s)$. Another equation is required in order to obtain a solution. This additional equation can come from the relationship between terminal voltage v_a, and load current - which is the current i_a. Thus:

$$v_a = R_L i_a + L_L \frac{di_a}{dt} \qquad \text{(IV-12)}$$

(IV-12) yields the transformed equation:

$$v_a(s) = (R_L + L_L s)\, I_a(s) \qquad \text{(IV-13)}$$

Equations (IV-11) and (IV-13) can be solved for either I_a or v_a. V_a is probably the variable of interest so we will solve for $v_a(t)$. Thus:

$$v_a(s) = \frac{\begin{vmatrix} \frac{v_a(0)}{s} & r_a(1 + T_a s) \\ 0 & -R_L(1 + T_L s) \end{vmatrix}}{\begin{vmatrix} 1 & r_a(1 + T_a s) \\ 1 & -R_L(1 + T_L s) \end{vmatrix}} = \frac{R_L(1 + T_L s)\, v_a(0)}{s[R_L(1 + T_L s) + r_a(1 + T_a s)} \qquad \text{(IV-14)}$$

where:

$$T_a = \frac{L_a}{r_a} \; ; \; T_L = \frac{L_L}{R_L} \qquad \text{(IV-15)}$$

which is:

$$v_a(s) = \frac{R_L v_a(0)}{(R_L + r_a)} \left\{ \frac{(1 + T_L s)}{s\,(1 + T\, s)} \right\} \qquad \text{(IV-16)}$$

where:

$$T = \frac{L_L + L_a}{r_a + R_L} \qquad \text{(IV-17)}$$

We can apply the Initial Value and Final Value Theorems to (IV-16) to determine the terminal voltage at the instant of closing the switch and after the machine reaches a steady state condition. Thus,

$$\underset{t\to 0}{\text{Limit}}\ V_a(t) = \underset{t\to\infty}{\text{Limit}}\ s\ V_a(s) = \underset{s\to\infty}{\text{Limit}} \left\{\frac{\frac{1}{s} + T_L}{\frac{1}{s} + T}\right\} \frac{R_L}{(R_L + r_a)}\ V_a(0) \qquad \text{(IV-18)}$$

or:

$$\underset{t\to 0}{\text{limit}}\ V_a(t) = \left(\frac{T_L}{T}\right) \frac{R_L}{R_L + r_a}\ V_a(0) = \frac{L_L}{L_L + L_a}\ V_a(0) \qquad \text{(IV-19)}$$

Note that $V_a(0)$ is the terminal voltage before load is switched. $\underset{t\to 0}{\text{Limit}}\ V_a(t)$ is the instantaneous voltage immediately after the load is switched. For the steady state value:

$$\underset{t\to\infty}{\text{Limit}}\ V_a(t) = \underset{s\to 0}{\text{Limit}}\ s\ V_a(s) = \underset{s\to 0}{\text{limit}} \frac{R_L}{R_L + r_a}\ V_a(0) \left\{\frac{1 + T_L s}{1 + T s}\right\} \qquad \text{(IV-20)}$$

or:

$$\underset{t\to\infty}{\text{Limit}}\ V_a(t) = \frac{R_L}{R_L + r_a}\ V_a(0) = V_a \qquad \text{(IV-21)}$$

where V_a is the steady state terminal voltage. IF (IV-21) is rearranged, we have

$$V_a = - r_a \frac{V_a}{R_L} + V_a(0) \qquad \text{(IV-22)}$$

Recognize that the steady state current i_a is steady state voltage divided by load resistance. Thus:

$$i_a = \frac{V_a}{R_L} \qquad \text{(IV-23)}$$

If (IV-23) is substituted in (IV-22), we have the steady state terminal voltage characteristic of the separately excited shunt generator in terms of load current, no load voltage, and armature resistance. Thus,

$$V_a = - r_a I_a + V_a(0) \qquad \text{(IV-24)}$$

This is a linear relationship and plots as shown in Figure IV-4.

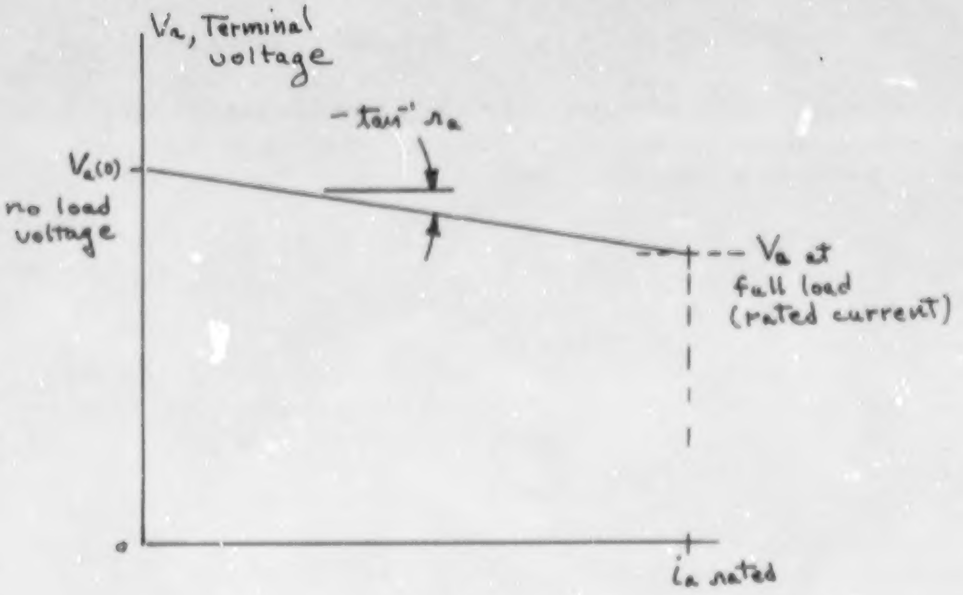


Figure IV-4. Steady State Voltage-Current Characteristic, Separately Excited Shunt Generator

A measure of the ability of a generator to maintain nearly constant voltage is the so called "voltage regulation". By definition;

$$\% \text{ Voltage Regulation} = \frac{V_a(0) - V_a(\text{full load})}{V_a\ (\text{full load})} \times 100$$

It should be emphasized that the above is representative of the steady state relationship between terminal voltage and load current. It may be that the terminal voltage as a function of time, after load is switched, is desired. This information can be obtained from the inverse of (IV-16). Thus;

$$V_a(t) = \mathcal{L}^{-1}(V_a(s)) = \frac{R_L}{R_L + r_a} V_a(0) \cdot \left\{1 + \frac{T_L - T}{T} \epsilon^{-t/T}\right\} \qquad \text{(IV-25)}$$

$$= V_a(0) \frac{R_L}{R_L + r_a} \cdot \left\{1 - \epsilon^{-t/T} + \frac{L_L/R_L}{\frac{L_L + L_a}{R_L + r_a}} \epsilon^{-t/T}\right\} \qquad \text{(IV-26)}$$

$$= \frac{V_a(0)}{1 + \frac{r_a}{R_L}} (1 - \epsilon^{-t/T}) + \frac{V_a(0)}{1 + \frac{L_a}{L_L}} \epsilon^{-t/T} \qquad \text{(IV-27)}$$

The response described in (IV-27) can be plotted as a function of time by plotting each term and then graphically summing. This is as shown in Figure (IV-5).

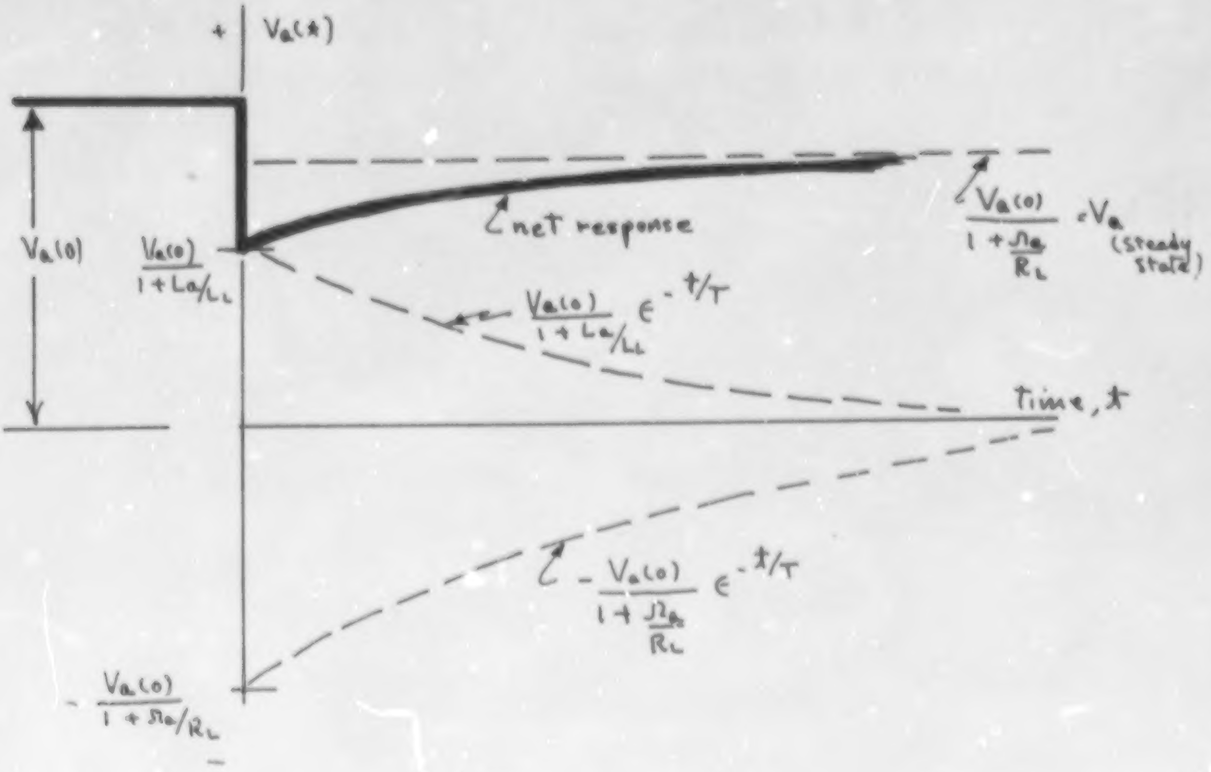


Figure IV-5. $V_a(t)$ for the Separately Excited Shunt Generator

We can summarize the response by noting that the voltage dips instantaneously and then varies in an exponential fashion to the final steady state value determined by the load resistance. The initial dip would occur in spite of a voltage regulator which would sense changes in output or terminal voltage and then act to change V_f or r_f in order to restore V_a to its prior value. The time constant associated with the voltage variation is determined by the total inductance and resistance in the armature circuit.

To investigate the transient behavior of the shaft power input to the generator, refer to (IV-7) and (IV-10):

$$P_{mech} = -\omega i_a i_f L_{df} = -\omega i_a L_{df}\left(\frac{V_a(0)}{\omega L_{df}}\right) = i_a V_a(0) \qquad \text{(IV-28)}$$

or

$$P_{mech}(s) = -V_a(0)\, I_a(s) \qquad \text{(IV-29)}$$

We can use a value of $I_a(s)$ as determined from (IV-11) and (IV-13). Then:

$$P_{mech}(s) = -V_a(0) \frac{\begin{vmatrix} 1 & \frac{V_a(0)}{s} \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & (r_a + L_a s) \\ 1 & -(r_L + L_L s) \end{vmatrix}} \qquad \text{(IV-30)}$$

$$= -\frac{V_a^{\,2}(0)}{(r_a + R_L)} \left\{ \frac{1}{s\,(1 + T\,s)} \right\} \qquad \text{(IV-31)}$$

where:

$$T = \frac{L_L + L_a}{R_L + r_a} \qquad \text{(IV-32)}$$

Applying the Initial Value Theorem:

$$\lim_{t \to 0} P_{mech}(t) = \lim_{s \to \infty} s\, P_{mech}(s) = 0 \qquad \text{(IV-33)}$$

This should not be surprising since the machine is at no load, electrically, and we are not including mechanical losses (friction and windage). Therefore P_{mech} at $t = 0$ is zero.

Applying the Final Value Theorem yields:

$$\lim_{t \to \infty} P_{mech}(t) = \lim_{s \to 0} s\, P_{mech}(s) = \frac{V_a^{\,2}(0)}{r_a + R_L} \qquad \text{(IV-34)}$$

Again, this is a reasonable answer for a physical reasoning standpoint. Since field current and speed do not change, the induced voltage remain the same as the no load terminal voltage and the steady state power input will be the total power dissipated as heat in r_a and R_L.

From the inverse transform of (IV-41), we have

$$P_{mech}(t) = - \frac{V_a^2(0)}{r_a + R_L} (1 - \varepsilon^{-t/T}) \qquad \text{(IV-35)}$$

The minus sign means mechanical power into the machine.

An example will serve to illustrate the use of the equations and also conversion from actual values to per unit values.

Example

A D.C. machine, operating as a generator, has the following rating: 200 Kw, 250 volts, 900 rpm.

Test results yield the following:

field resistance, r_f = 33.7 ohms

q axis armature resistance, r_a = 0.0125 ohms

field inductance, L_f = 25 henry

q axis armature inductance, L_a = 0.008 henry

When driven at rated speed, no load, an emf appears across the q axis armature winding equal to 22.8 volts per field ampere of excitation. In this problem, the generator is being driven at 1500 rpm. At $t = 0$, a series load of 0.313 ohms resistance, R_L, and 1.62 henry inductance, L_L, is switched onto the q axis armature winding. Prior to $t = 0$, the field winding has 230 volts impressed upon it and the current in the field has reached steady state. Find $V_a(t)$.

The value of L_{df} can be determined from the no load test information by using equation (IV-5):

$$V_a + r_a i_a + L_a \frac{di_a}{dt} = \omega L_{df} i_f \qquad \text{(IV-36)}$$

For:

$$i_a = 0$$

$$L_{df} = \frac{V_a}{\omega\, i_f} \qquad \text{(IV-37)}$$

Now, $\frac{V_a}{i_f} = 22.8$ when $\omega = \frac{900}{60} \times 2\pi = 94.4$ rad/sec. From which

$$L_{df} = \frac{22.8}{94.4} = 0.242 \text{ henry}$$

Using actual values, and substituting the values in (IV-17) and (IV-27) yields:

$$T = \frac{L_L + L_a}{R_L + r_a} = \frac{1.62 + 0.008}{0.313 + 0.0125} = 5.0$$

$$V_a(t) = \frac{V_a(0)}{1 + \frac{r_a}{R_L}}(1 - \varepsilon^{-t/T}) + \frac{V_a(0)}{1 + \frac{L_a}{L_L}} \varepsilon^{-t/T} \qquad \text{(IV-27)}$$

For $t < 0$, $V_f = 230$ and $i_f = \frac{230}{33.7} = 6.82a$

$$V_a(0) = \omega L_{df} i_f = (\frac{1500 \times 2\pi}{60})(0.242)(6.82) = 259.1 \qquad \text{(IV-38)}$$

$$1 + \frac{r_a}{R_L} = \frac{1 + 0.0125}{0.313} = 1.0399$$

$$1 + \frac{L_a}{L_L} = \frac{1 + 0.008}{1.62} = 1.0049$$

$$V_a(t) = 249.16 \, (1 - \varepsilon^{-t/5.0}) + 257.84 \, \varepsilon^{-t/5.0} \qquad \text{(IV-39)}$$

The initial value of voltage, after the switch is closed is 258 volts. The steady state value, for $t > 3T$, $t > (3)(5)$ is 249.

If we worked this in per unit, we would proceed as follows: First, choose a voltampere base of 200,000 watts, a voltage base of 250 volts, and a base angular velocity of 1.0. Then

$$I_b = \frac{200,000}{250} = 800$$

$$R_b = \frac{250}{800} = 0.3125 = Z_b$$

$$L_b = \omega_b Z_b = 1.0 \times 0.3125 = 0.3125$$

Converting the various actual values for the machine to per unit values yields

at t = 0, $$e_f = \frac{230}{250} = 0.92 \text{ pu}$$

$$r_f = \frac{33.7}{0.3125} = 107.84 \text{ pu}$$

$$L_f = \frac{25}{0.3125} = 80 \text{ pu}$$

$$i_f(0) = \frac{0.92}{107.84} = 0.0085$$

$$r_a = \frac{0.0125}{0.3125} = 0.04$$

$$R_L = \frac{0.313}{0.3125} = 1.0$$

$$L_a = \frac{0.008}{0.3125} = 0.0256$$

$$L_L = \frac{1.62}{0.3125} = 5.184$$

$$900 \text{ rpm} = 30\pi \text{ rad/sec} = 30\pi \text{ pu}$$

$$1500 \text{ rpm} = \frac{15}{9} \times 30\pi = 50\pi \text{ pu}$$

An open circuit, at rated speed

$$V_a(0) = 22.8 \text{ volts/field ampere}$$

and, from (IV-37)

$$L_{df} = \frac{V_a}{\omega i_f} = \frac{22.8}{30\pi} = 0.242 \text{ henry}$$

$$L_{df} = \frac{0.242}{0.3125} = 0.7745 \text{ pu}$$

Using per unit values,

$$T = \frac{5.184 \times 0.0256}{1.0 + 0.04} = 5.0$$

$$V_a(0) = \omega L_{df} i_f$$

$$= (50\pi)(0.7745)(\frac{0.92}{107.84}) = 1.03735$$

$$1 + \frac{r_a}{R_L} = 1 + \frac{0.04}{1.0} = 1.04$$

$$1 + \frac{L_a}{L_L} = 1 + \frac{0.0256}{5.184} = 1.0049$$

$$V_a(t) \text{ pu} = \frac{1.03735}{1.04} (1 - \varepsilon^{-t/5.0}) + \frac{1.03735}{1.0049} \varepsilon^{-t/5.0}$$

$$= 0.99745 (1 - \varepsilon^{-t/5.0}) + 1.03229 \varepsilon^{-t/5.0} \qquad \text{(IV-40)}$$

The value of $V_a(t)$, in per unit, at $t = 0$, is 1.03229. This is an actual value of (1.03229)(250) = 258.

The values of $V_a(t)$, in per unit, at values [illegible] > (3)(5) is 0.99745. This is an actual value of 249.

These values obtained by calculation in per unit agree with those obtained by calculation using actual values.

The voltage before load was switched and while the machine is running at 1500 rpm is:

$$V_a(0) = \frac{(1500)(2\pi)}{60} (0.242)(\frac{230}{33.7}) \text{ for } t < 0$$

$$= 259$$

A plot of $V_a(t)$ for this example is shown in Figure IV-6.

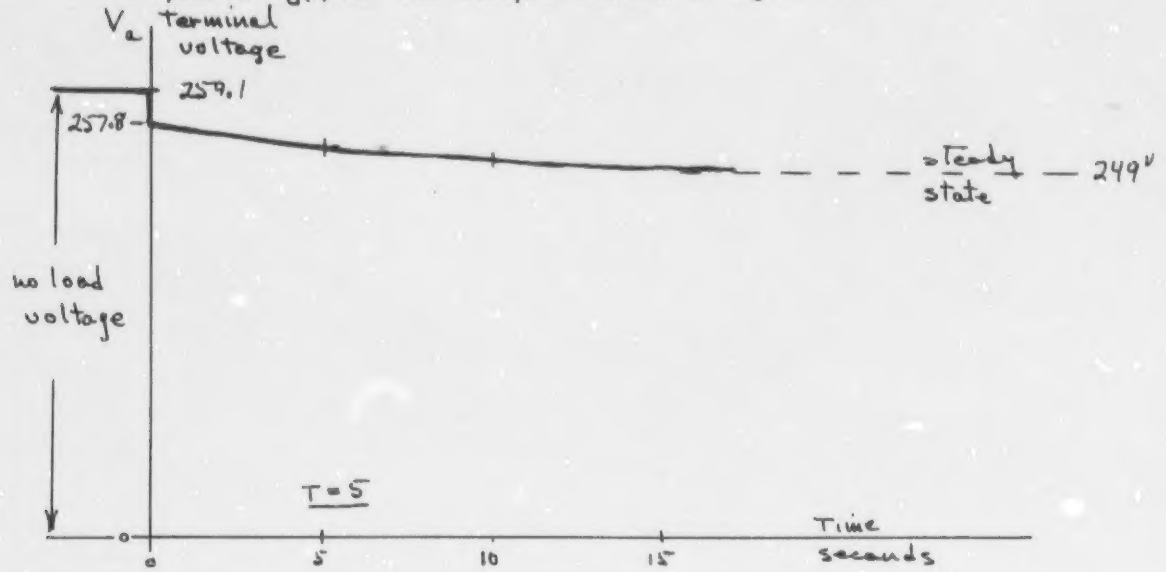
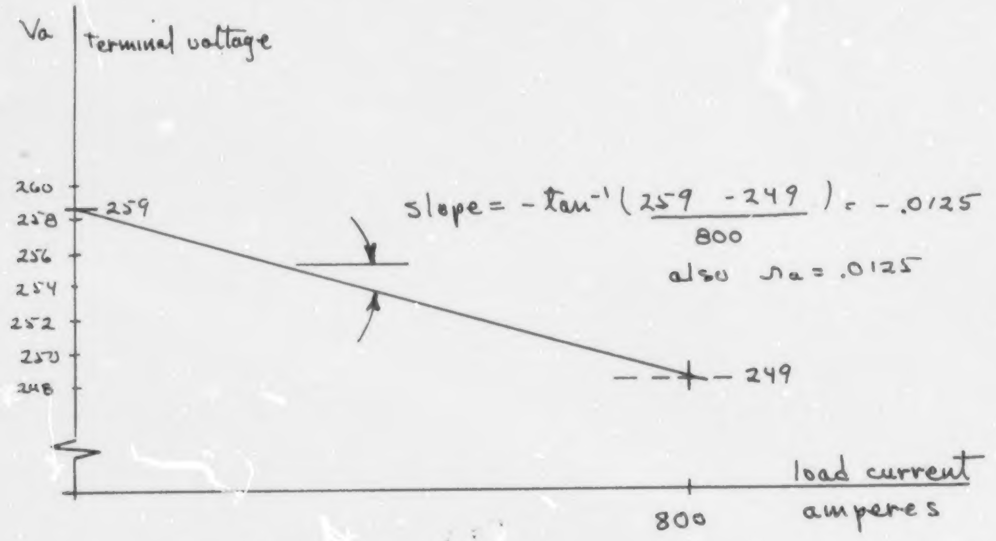


Figure IV-6. Terminal Voltage as a Function of Time

The voltage regulation can be calculated as (since 1.0 pu load was switched on):

$$\% \text{ voltage regulation} = + \left(\frac{259 - 249}{249}\right) 100 = 4.02\%$$

The steady state voltage-load characteristic for this example is shown (assuming linearity) in Figure IV-7.



$\text{slope} = -\tan^{-1}\left(\frac{259 - 249}{800}\right) = -.0125$

also $r_a = .0125$

Figure IV-7. Steady State Terminal Voltage as a Function of Load Current

IV.3 **THE SHUNT CONNECTED, SEPARATELY EXCITED, D.C. MOTOR:** This motor is widely used in industry because of the ease with which the speed-torque characteristic of the motor can be varied. Schematically, the windings are connected as shown in Figure (IV-8).

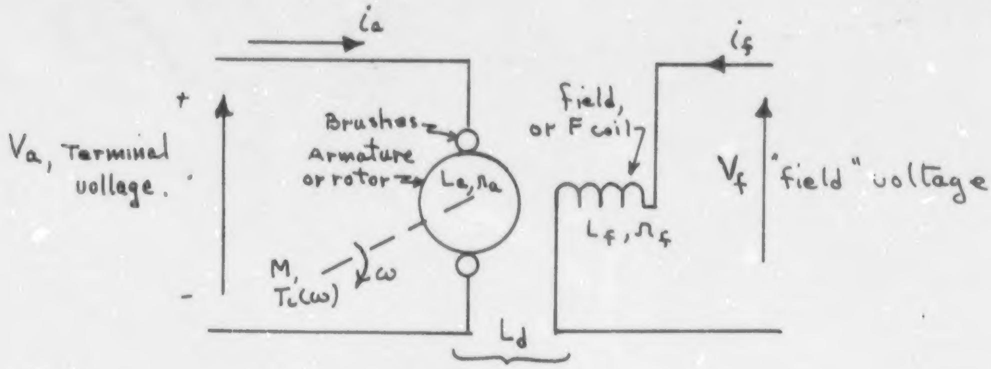

i_a
i_f
+
V_a, Terminal voltage.
-
Brushes
Armature or rotor
L_a, r_a
Field, or F coil
V_f "field" voltage
L_f, r_f
M, $T_L(\omega)$
ω
L_d

Figure IV-8. Separately Excited D.C. Shunt Motor

Our analysis is concerned with the speed variation with load torque. We will assume the mechanical losses (friction and windage) of the motor are negligible (or they could be included with the load torque requirement). The values of r_f and r_a include the resistance in each of the windings as well as any external resistance we might add in order to obtain a desired characteristic. The moment of inertia, J, includes the total moment of inertia connected to the shaft, i.e., for both the motor armature and load. We can mathematically describe the motor as follows:

$$V_f = r_f i_f + L_f \frac{di_f}{dt} \tag{IV-41}$$

$$V_a = r_a i_a + L_a \frac{di_a}{dt} + \omega\, L_{df} i_f \tag{IV-42}$$

$$L_{df} i_f i_a = T_L(\omega) + J \frac{d\omega}{dt} \tag{IV-43}$$

In order to proceed with the analysis, initial conditions are required. The initial conditions required are the values of i_f, ω, i_a, etc. at $t = 0$. At $t = 0$, the machine is operating in a steady state condition. Therefore, all time derivatives are zero. From (IV-41, -42, -43) denoting $T_L(\omega)$ at $t = 0$ as $T_L(0)$, ω at $t = 0$ as $\omega(0)$ etc.

$$V_f = r_f i_f (0) \ ; \quad i_f(0) = V_f/r_f \qquad \text{(IV-44)}$$

$$T_L(0) = L_{df} i_f(0)\ i_a(0) \qquad \text{(IV-45)}$$

$$i_a(0) = \frac{T_L(0) r_f}{L_{df} V_f} \qquad \text{(IV-46)}$$

$$\omega (0) = \frac{V_a r_f}{L_{df} V_f} - \frac{r_a T_L(0)\ r_f^2}{L_{df}^2\ e_f^2} \qquad \text{(IV-47)}$$

In this analysis, we will investigate only variations resulting from changing mechanical load. Of course, the armature current will vary with time, but the field current will remain constant (no coupling between armature and field). We need only solve the latter two of the three equations (IV-41, -42, -43). Taking the transform of these, we have:

$$\frac{V_a}{s} = r_a \ (1 + \frac{L_a}{r_a} s) I_a(s) - L_a i_a(0) + L_{df}\, i_f\, \omega (s) \qquad \text{(IV-48)}$$

$$T_L(s) = L_{df} i_f\ I_a(s) - J\ s\, \omega (s) + J\, \omega (0) \qquad \text{(IV-49)}$$

In order to obtain behavior characteristics for the simplest case, consider that the machine is at no load at $t = 0$ and that armature inductance is negligible (an often used assumption) and that a step type torque load is impressed. For no load, from (III-106);

$$T_e = 0 = i_q \lambda_d = i_d \lambda_d = i_a L_{df} i_f \qquad \text{(IV-50)}$$

or:

$$i_a(0) = 0 \tag{IV-51}$$

and:

$$T_L(\omega) = \frac{T_L}{s} \tag{IV-52}$$

From (III-101) and (III-103); (with $i_a(0) = 0$)

$$V_a = \omega(0)\, L_{df} i_f = \omega(0) L_{df} \frac{V_f}{r_f} \tag{IV-53}$$

or

$$\omega\,(0) = \frac{V_a\, r_f}{V_f\, L_{df}} \tag{IV-54}$$

Using (IV-51), (IV-52) and (IV-54) in (IV-48) and (IV-49) yields:

$$r_a I_a(s) + L_{df} \frac{V_f}{r_f}\, \omega(s) = \frac{V_a}{s} \tag{IV-55}$$

$$L_{df} \frac{V_f}{r_f} - J\, s\, \omega(s) = -\, J\, \omega(0) + \frac{T_L}{s} \tag{IV-56}$$

We can solve for $\omega\,(s)$ as follows:

$$\omega\,(s) = \frac{\begin{vmatrix} r_a & \dfrac{V_a}{s} \\ L_{df}\dfrac{V_f}{r_f} & -\, J\, \omega(0) + \dfrac{T_L}{s} \end{vmatrix}}{\begin{vmatrix} r_a & L_{df}\dfrac{V_f}{r_f} \\ L_{df}\dfrac{V_f}{r_f} & -\, J\, s \end{vmatrix}} \tag{IV-57}$$

$$= \frac{r_a\, J\, \omega(0) + \dfrac{V_a\, L_{df} V_f}{r_f\, s} - \dfrac{T_L\, r_a}{s}}{r_a\, J\, s + \left(\dfrac{L_{df} V_f}{r_f}\right)^2} \tag{IV-58}$$

Applying the Final Value Theorem yields:

$$\omega_{ss} = \lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\,\omega(s) = \frac{V_a\, r_f}{V_f\, L_d} - \frac{T_L\, r_a\, r_f^2}{V_f^2\, L_{df}^2} \qquad \text{(IV-59)}$$

Applying the Initial Value Theorem yields:

$$\omega(0) = \lim_{t\to 0} \omega(t) = \lim_{s\to\infty} s\,\omega(s) = \omega(0) \qquad \text{(IV-60)}$$

We will define an apparent time constant, T, as:

$$T = \frac{r_a\, J\, r_f^2}{L_{df}^2\, V_f^2} \qquad \text{(IV-61)}$$

with this defined value of T, and the value ω_{ss} from (IV-59), we can express $\omega(s)$ in (IV-58) as

$$\omega(s) = \omega_{ss} \left\{ \frac{T \frac{\omega(0)}{\omega_{ss}} s + 1}{s(T s + 1)} \right\} \qquad \text{(IV-62)}$$

The inverse transform of (IV-62) yields $\omega(t)$ as:

$$\omega(t) = \omega_{ss}(1 - \varepsilon^{-t/T}) + \omega(0)\, \varepsilon^{-t/T} \qquad \text{(IV-63)}$$

This solution is plotted in Figure (IV-9).

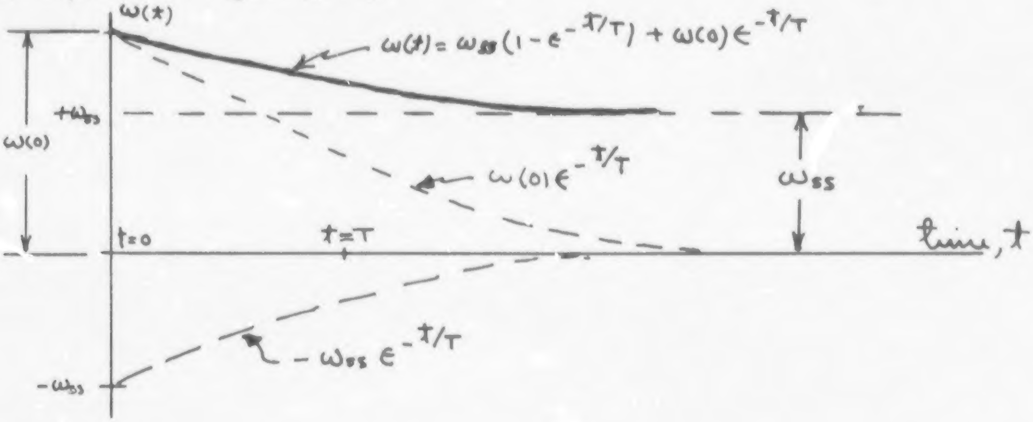


Figure IV-9. $\omega(t)$ for Step Torque Load on a Separately Excited Shunt Motor

From Figure (IV-9) the speed change with time is exponentially decreasing to its steady state value. The time constant associated with the machine is a function of armature circuit resistance, moment of inertia, the field resistance, field voltage and mutual inductance of the d axis.

The steady state speed-torque characteristis is of extreme interest and the fact that it can be easily varied, or altered, in a D.C. shunt motor accounts for the widespread useage and popularity of this motor, in spite of the necessity for providing a source of DC power and the increased maintenance problems (due to brushes, mainly) associated with the DC machine.

From (IV-59), the variables which affect the speed torque relationship are:

1. the armature voltage, V_a
2. the armature circuit resistance, r_a
3. the field current, changed by changing V_f and/or r_f.

Replacing the ratio V_f to r_f by the field current, i_f, (IV-59) becomes:

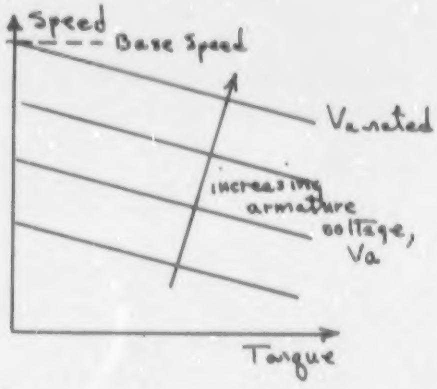
$$\omega_{ss} = (\frac{1}{L_{df}}) \frac{V_a}{i_f} - (\frac{1}{L_{df}})^2(\frac{r_a}{{i_f}^2})T_L \qquad \text{(IV-64)}$$

In order to examine the various possibilities we will examine (IV-64) under various conditions. For example, consider all external resistance in the field circuit removed and rated voltage applied to the field circuit. i_f is constant and (IV-64) becomes:

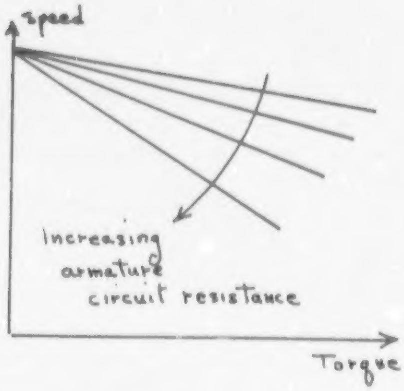
$$\omega_{ss} = \frac{V_a}{K} - \frac{r_a}{K^2}T_L \qquad \text{(IV-65)}$$

This is the equation of a straight line when steady state speed is plotted against load torque. A family of curves are obtainable, as shown in Figure IV-10(a) if the voltage applied to the armature circuit is varied from zero up to rated value. The slope of the characteristic (for fixed V_a) is proportional to the armature circuit resistance. This is shown in Figure IV-10(b), which depicts the situation where i_f and V_a are constant but various values of armature resistance, r_a, are used.

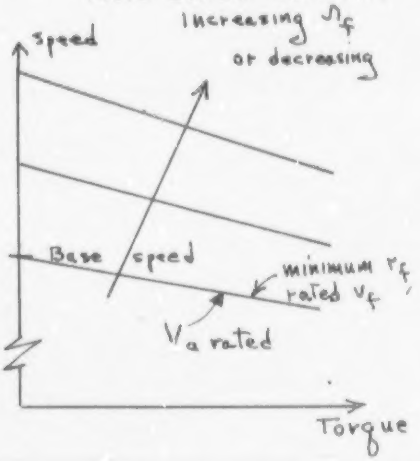
Now, consider that rated armature voltage, V_a, and rated field voltage V_f are applied and that all external resistance from the field circuit is removed. The resulting speed is the so called "base speed" of the motor. In order to increase the speed above the base speed value, the field current, i_f, must be reduced - either by increasing field circuit resistance, r_f, or by reducing the voltage applied to the field, V_f. The resulting speed torque characteristics are shown in Figure IV-10(c).



(a) Speed-Torque Variation Below "Base Speed". Full Field Current Applied and Maintained Constant and Armature Resistance Constant.



(b) Speed-Torque Variation by Changing r_a. Applied Armature Voltage and Field Current Constant.



(c) Speed Control Above Base Speed. Rated V_a and varying V_f or r_f.

Figure IV-10. Shunt Motor Speed Control

Voltage regulation was previously defined for the generator. In the same fashion, we can define percentage speed regulation. Thus:

$$\% \text{ Speed Regulation} = \frac{\left(\omega(0) - \omega_{\text{full load}}\right) \times 100}{\omega_{\text{full load}}} \qquad \text{(IV-66)}$$

Note that speed regulation increases with increasing r_a when using fixed field, variable armature voltage control and with increasing $\omega(0)$, (decreasing i_f) when using fixed armature circuit voltage and resistance with variable field excitation control.

In addition to "base speed" as defined above, machines have a "top speed" limitation as determined by mechanical considerations, i.e., bearings, winding centrifugal stresses, etc.

Below base speed, the field current is constant at its maximum value. There is a maximum value of armature current possible because of the heating losses in the armature winding. Since, in steady state, the torque is proportional to i_f and i_a the torque limitation is constant at the continuous maximum value for speeds from zero to base speed. With constant torque, the power rating is linearly proportional to speed.

Figure IV-11 depicts the torque and power limitations over the same speed range. When base speed is reached, higher speed is obtained only by decreasing the field current. For purposes of illustration, with negligible armature resistance, from (IV-64):

$$\omega \propto \frac{1}{i_f} \qquad \text{(IV-67)}$$

and from (III-103) and (III-106)

$$T \propto i_f \qquad \text{(IV-68)}$$

From (IV-67) and (IV-68)

$$T \propto \frac{1}{\omega} \qquad \text{(IV-69)}$$

Thus, the torque-speed limitation between base speed and top speed is hyperbolic, as shown in Figure IV-11. If both sides of (IV-69) are multiplied by ω, it can be seen that the power is proportional to a constant for the speed range obtained by field control, as shown in Figure IV-11.

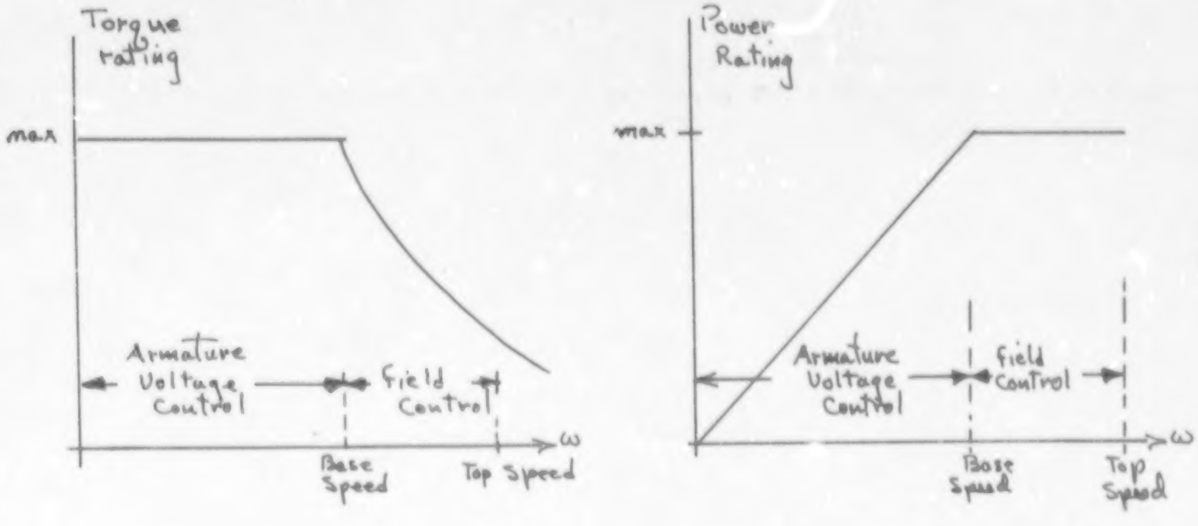


Figure IV-11. Torque and Power Limitations - DC Shunt Motor

IV.4 BLOCK DIAGRAMS: In system analysis it is often helpful to have block diagram representation of entities comprising the system. To illustrate the procedures to be followed in the derivation of the block diagram, we will go through the procedure for the shunt connected d.c. motor analyzed in Section IV.3.

Block diagram representation is simply combining the various transfer functions, derived from the mathematical model of the device being represented, to represent the system as a whole. A transfer function cannot include initial conditions but initial conditions can be incorporated into the block diagram. To insure the absence of initial conditions, we will deal with changes in the various variables and add the initial conditions after deriving the representation for the changes in the variable. If all changes result from something that takes place at $t = 0$, the changes do not have initial conditions and transfer function representation is valid.

The block diagram representation is often obtained as a prelude to computer representation of the device or system. Since both field voltage and armature voltage can alter the shunt connected DC motor behavior, the representation will be based on all three ofthe describing equations, (IV-41, -42 and -43).

These equations are:

$$V_f = r_f i_f + L_f \frac{di_f}{dt} \qquad \text{(IV-41)}$$

$$V_a = r_a i_a + L_a \frac{di_a}{dt} + \omega L_{df} i_f \qquad \text{(IV-42)}$$

$$L_{df} i_f i_a = T_L + J \frac{d\omega}{dt} \qquad \text{(IV-43)}$$

Consider that the variables, V_f, i_f, V_a, i_a, T_L and ω in these equations represent the steady state value prevailing before $t = 0$ plus the portion of the variable that changes after $t = 0$. Thus:

$$i_f = i_f(0) + \Delta i_f \tag{IV-70}$$

$$V_f = V_f(0) + \Delta V_f \tag{IV-71}$$

$$i_a = i_a(0) + \Delta i_a \tag{IV-72}$$

$$V_a = V_a(0) + \Delta V_a \tag{IV-73}$$

$$\omega = \omega(0) + \Delta\omega \tag{IV-74}$$

$$T_L = T_L(0) + \Delta T_L \tag{IV-75}$$

(IV-41) becomes, using (IV-70):

$$V_f(0) + \Delta V_f = r_f(i_f(0) + \Delta i_f) + L_f \frac{d}{dt}(i_f(0) + \Delta i_f) \tag{IV-76}$$

Also, the steady state relationship, for $t < 0$ prevails, i.e.,

$$V_f(0) = r_f\, i_f(0) \tag{IV-77}$$

Subtracting (IV-77) from (IV-76) and finding the Laplace Transform yields:

$$\Delta V_f = r_f \Delta i_f + L_f s\, \Delta i_f \tag{IV-78}$$

$$\Delta V_f = r_f \left(1 + \frac{L_f}{r_f} s\right) \Delta i_f \tag{IV-79}$$

It is current, i_f, that provides excitation flux and is thus the variable of interest in our modeling and block diagram. (IV-79) should be solved for Δi_f to obtain the transfer function as shown in Figure IV-12.

$$\Delta i_f = \frac{1}{r_f\left(1 + \frac{L_f}{r_f} s\right)} \Delta V_f \tag{IV-80}$$

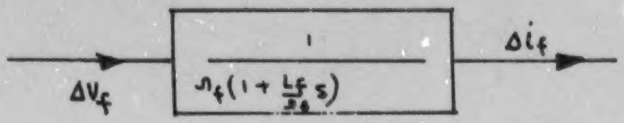


Figure IV-12. Block Diagram Representation (Transfer Function) for (IV-80).

Equation (IV-42) can be subjected to the same manipulation as (IV-41). Thus:

$$V_a(0) + \Delta V_a = r_a(i_a(0) + \Delta i_a) + L_a \frac{d}{dt} (i_a(0) + \Delta i_a) +$$

$$+ (\omega(0) + \Delta\omega) L_{df}(i_f(0) + \Delta i_f) \tag{IV-81}$$

$$V_a(0) = r_a i_a(0) + \omega(0) L_{df} i_f(0) \tag{IV-82}$$

Subtracting (IV-82) from (IV-81) and transforming, yields:

$$\Delta V_a = r_a \Delta i_a + L_a s \Delta i_a + L_{df}(i_f(0)\Delta\omega + \Delta i_f \omega(0) + \Delta i_f \Delta\omega) \tag{IV-83}$$

Note that $\Delta i_f \Delta\omega$ is a second order effect which will be neglected.

In determining which variable, or variables, will be inputs and which will be outputs in the block diagram representation of (IV-83) we reason that V_a is an input and that in the overall diagram ω is an output. An output is available to be used as an input if necessary (feedback). Some experience is helpful in deciding how the block diagram is to be arranged. In this situation, we will use, as a first effort at least, the arrangement shown in (IV-84) and Figure IV-13.

$$\Delta i_a = \frac{\Delta V_a - L_{df} i_f(0) \Delta\omega - L_{df}\omega(0)\Delta i_f}{r_a(1 + \frac{L_a}{r_a} s)} \tag{IV-84}$$

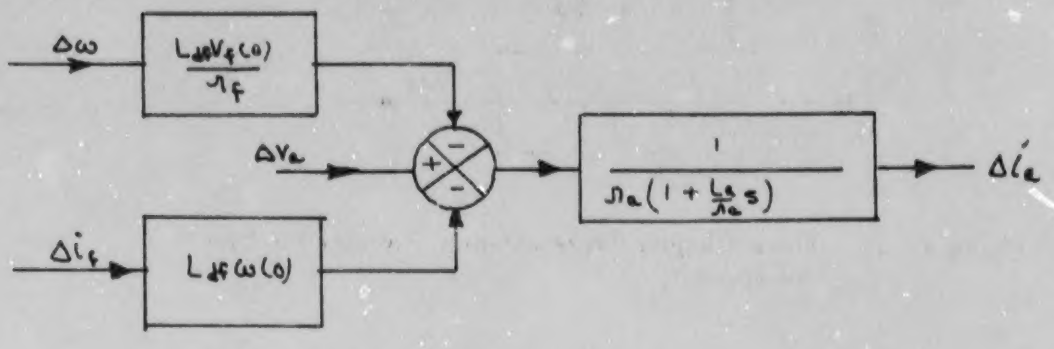

Δω
$\frac{L_{df}V_f(0)}{r_f}$
ΔV_a
+
−
−
$\frac{1}{r_a(1+\frac{L_a}{r_a}s)}$
Δi_a
Δi_f
$L_{df}\omega(0)$

Figure IV-13. Block Diagram for (IV-84).

For (IV-43, applying the same procedures,

$$L_{df}(i_f(0) + \Delta i_f)(i_a(0) + \Delta i_a) = (T_L + \Delta T_L) + J \frac{d}{dt}(\omega(0) + \Delta\omega) \tag{IV-85}$$

and:

$$L_{df} i_a(0)\, i_f(0) = T_L \tag{IV-86}$$

From which;

$$\Delta\omega = \frac{1}{Js}[-\Delta T_L + L_{df} i_f(0)\Delta i_a + L_{df} i_a(0)\Delta i_f] \tag{IV-87}$$

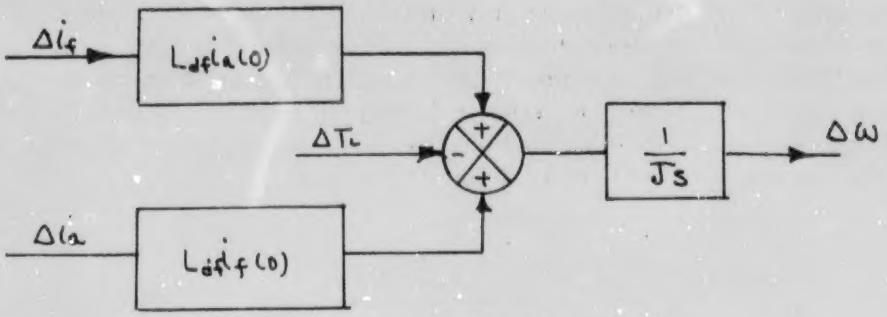

Δi_f
$L_{df}i_a(0)$
ΔT_L
+
−
+
$\frac{1}{Js}$
Δω
Δi_a
$L_{df}i_f(0)$

Figure IV-14. Block Diagram for (IV-87).

We are now in a position to formulate the Block Diagram for the complete system by combining Figure IV-12, IV-13, and IV-14. The result is as depicted in Figure IV-15 and Figure IV-16.

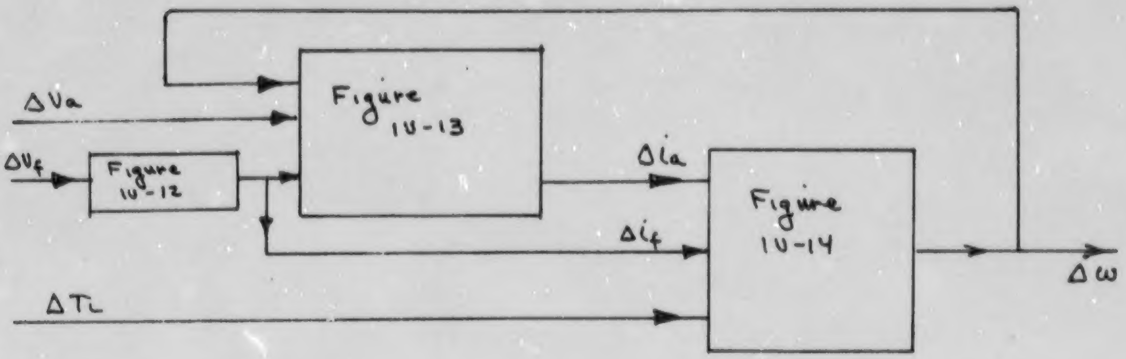

ΔV_a
ΔV_f
Figure IV-12
Figure IV-13
Δi_a
Δi_f
Figure IV-14
ΔT_L
$\Delta \omega$

Figure IV-15. Connection Diagram for Figures IV-12, -13, and -14.

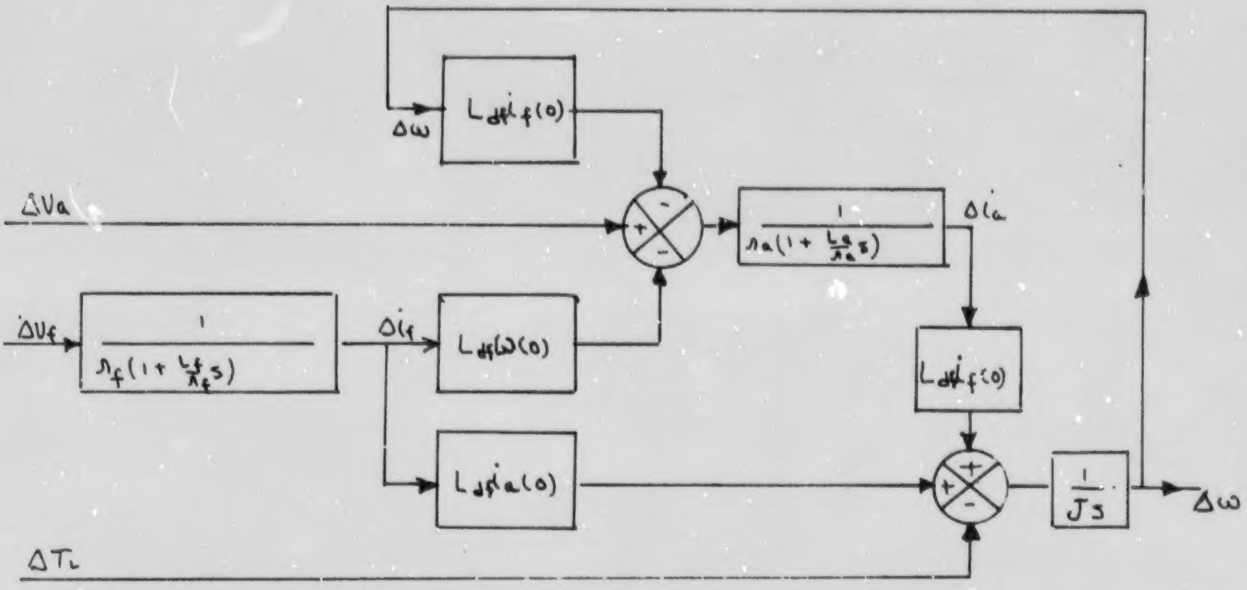

$\Delta \omega$
$L_{af} i_f(0)$
ΔV_a
+
-
-
$\frac{1}{r_a(1+\frac{L_a}{r_a}s)}$
Δi_a
ΔV_f
$\frac{1}{r_f(1+\frac{L_f}{r_f}s)}$
Δi_f
$L_{af} \omega(0)$
$L_{af} i_f(0)$
$L_{af} i_a(0)$
+
+
-
$\frac{1}{Js}$
$\Delta \omega$
ΔT_L

Figure IV-16. Block Diagram for Shunt Connected DC Motor

If the field circuit parameters V_f, i_f, etc. are maintained constant, the Block Diagram of Figure IV-17 is considerably simplified and is as shown in Figure IV-17.

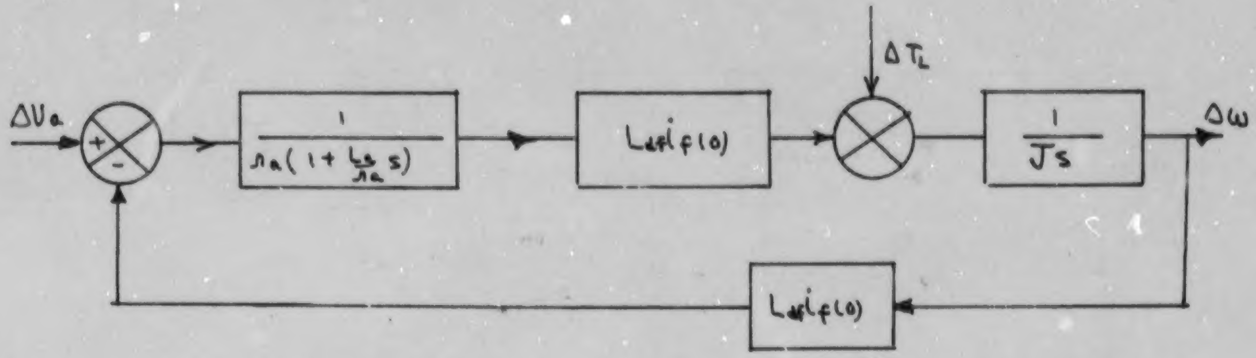


Figure IV-17. Block Diagram for Shunt Connected DC Motor with Constant Excitation.

From the Block Diagram, it can be seen that the shunt motor comprises a negative feedback system.

IV.5 THE SERIES D.C. MOTOR: The series D.C. motor has rather wide usage in applications requiring maximum torque near zero (or stall) speed - such as in traction systems, etc. Because of the nonlinearity inherent in the analysis it will be instructive to examine the method of analysis in detail.

The series machine has a field winding which is connected in series with the armature circuit. The applicable equations are:

$$T_e = T_L(\omega) + J\frac{d\omega}{dt} \qquad \text{(IV-88)}$$

$$V_a = r_a i_a + L_a \frac{di_a}{dt} + \omega\lambda_d \qquad \text{(IV-89)}$$

$$\lambda_d = L_{df} i_a \qquad \text{(IV-90)}$$

$$T_e = \lambda_d i_a = L_{df} i_a^2 \qquad \text{(IV-91)}$$

The i_a^2 term in these equations results in the system of equations being non linear. A technique for dealing with this will be presented. First, we will examine the steady state characteristic of the motor. In the steady state, neglecting mechanical losses;

$$T_L = L_{df} i_a^2 \tag{IV-92}$$

$$V_a = r_a i_a + \omega L_{df} i_a \tag{IV-93}$$

i_a can be eliminated from (IV-92) and (IV-93) yielding:

$$T_L = L_{df} \left(\frac{V_a}{r_a + \omega L_{df}} \right)^2 \tag{IV-94}$$

From (IV-94), as $\omega \to 0$

$$T_L \to L_{df} \frac{V_a^2}{r_a^2} = L_{df} i_a^2 \tag{IV-95}$$

In other words, T_L is proportional to the square of the current flowing when current is determined only by applied voltage and armature circuit resistance, i.e., no counter electromotive force, $\omega\lambda$, exists to limit current. Further, as $\omega \to \infty$, $T_L \to 0$.

The steady state characteristic is shown in Figure IV-18.

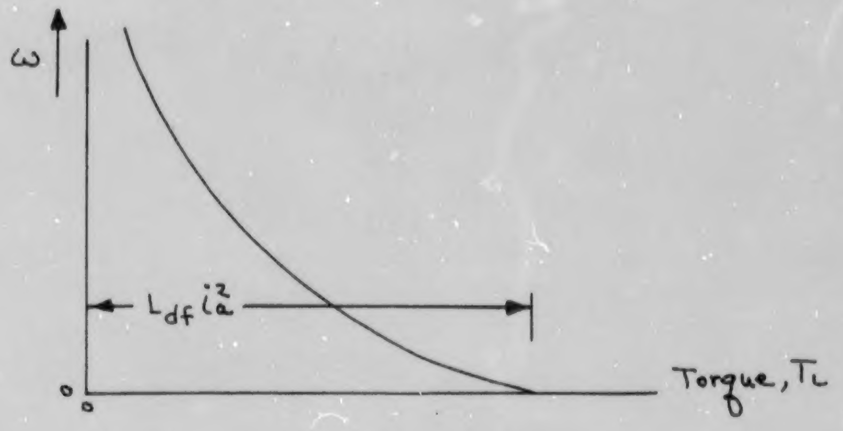


Figure IV-18. Series DC Motor Speed Torque Characteristic

Because of the excessively high speeds resulting as $T_L \rightarrow 0$, care must be exercised in the application aspects of series motors. Belt drives are never used because of the possibility of a belt failure and consequent unloading and overspeeding.

Because of the non linear differential equations which describe the DC series motor, (IV-88, -89, -90, -91) the dynamics of the motor are best studied using a computer. However, we can utilize the technique of "linearizing about an operating point" to eliminate the non linearity. This technique is only valid for extremely small changes in operating condition from the steady state operating point. It does yield considerable insight into the pattern of the dynamic behavior and does permit the application of linear analysis techniques to the solution of the describing equations.

In effect, we will consider the machine to be in the steady state condition with some load, $T_L(0)$. We will introduce a small change in load, ΔT_L and observe the oscillations which result. We will denote all the other variables in the same fashion. Thus, at steady state,

$$L_d i_a^2(0) = T_L(0) \tag{IV-96}$$

$$V_a(0) = r_a i_a(0) + L_{df} i_a(0)\,\omega(0) \tag{IV-97}$$

For small changes, (IV-88, -89, -90, -91) becomes:

$$L_{df}(i_a(0) + \Delta i_a)^2 = T_L(0) + \Delta T_L + J \frac{d}{dt}(\omega(0) + \Delta\omega) \tag{IV-98}$$

$$V_a(0) + \Delta V_a = r_a(i_a(0) + \Delta i_a) + L_a \frac{d}{dt}(i_a(0) + \Delta i_a) +$$

$$+ (\omega(0) + \Delta\omega)L_{df}(i_a(0) + \Delta i_a) \tag{IV-99}$$

Subtracting (IV-96) from (IV-98) and (IV-97) from (IV-99) taking the Laplace transform and neglecting the second order differential yields:

$$\Delta T_L = 2L_{df} i_a(0) \Delta i_a - J s \Delta\omega \tag{IV-100}$$

$$\Delta V_a = (r_a + \omega(0) L_{df}) \left\{ 1 + \frac{L_a}{r_a + \omega(0) L_{df}} s \right\} \Delta i_a +$$

$$+ L_{df} i_a(0) \Delta\omega \tag{IV-101}$$

In this analysis, assume that V_a remains constant, i.e., $\Delta V_a = 0$ and that the perturbation arises from ΔT_L. If ΔT_L is a small step input type perturbation

$$\Delta T_L = \frac{\Delta T}{s} \qquad \text{(IV-102)}$$

Also, we will denote:

$$\frac{L_a}{r_a + \omega(0) L_{df}} = T = \frac{L_a}{R_a} \qquad \text{(IV-103)}$$

where T is the "apparent" time constant of the armature circuit and R_a is the "apparent" armature circuit resistance.

We can solve for $\Delta\omega$ from:

$$= \frac{\begin{vmatrix} 2L_{df} i_a(0) & \frac{\Delta T}{s} \\ R_a(1 + T s) & 0 \end{vmatrix}}{\begin{vmatrix} 2L_{df} i_a(0) & - J s \\ R_a(1 + T s) & L_{df} i_a(0) \end{vmatrix}} = \frac{\Delta T \; R_a (1 + T s)}{s[J L_a s^2 + J R_a s + 2L_{df}^{2} i_a^{2}(0)} \qquad \text{(IV-104)}$$

$$= \frac{R_a}{2L_{df}^{2} i_a^{2}(0)} \Delta T \left[\frac{1 + T s}{s(s^2 \frac{J L_a}{2L_{df}^{2} i_a^{2}(0)} + \frac{J R_a}{2L_{df}^{2} i_a^{2}(0)} s + 1)} \right] \qquad \text{(IV-105)}$$

This is of the general form;

$$\Delta\omega = A \left[\frac{1 + T s}{s(\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1)} \right] \qquad \text{(IV-106)}$$

The inverse transform of (IV-106) is:

$$\Delta\omega = A'\left\{1 + \frac{1}{\sqrt{1-\xi^2}}\left(1 - 2T\xi\omega_n + T^2\omega_n^2\right)^{1/2}\varepsilon^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}\,t + \psi\right)\right\} \qquad \text{(IV-107)}$$

where:

$$\psi = \tan^{-1}\frac{T\omega_n\sqrt{1-\xi^2}}{1 - T\omega_n\xi} - \tan^{-1}\frac{\sqrt{1-\xi^2}}{-\xi} \qquad \text{(IV-108)}$$

It should be noted that an exact solution would not be meaningful since we linearized our equations and they are valid only as the incremental changes $\rightarrow 0$. However, we can determine the frequency of oscillations and the time constant associated with the decay of these oscillations.

Comparing (IV-105), (IV-106) and (IV-107) we can determine the following:

$$\omega_n = L_{df} i_a(0)\sqrt{\frac{2}{JL_a}} \qquad \text{(IV-109)}$$

= undamped natural angular velocity

and, since;

$$T_L(0) = L_{df} i_a^2(0) \qquad \text{(IV-110)}$$

(IV-109) can be expressed as:

$$\omega_n = \sqrt{\frac{2 L_{df} T_L(0)}{J L_a}} \qquad \text{(IV-111)}$$

and

$$\xi = \frac{\omega_n}{2}\,\frac{J R_a}{2L_{df}^2\, i_a^2(0)} = \frac{1}{T}\sqrt{\frac{L_a J}{8L_{df} T_L(0)}} \qquad \text{(IV-112)}$$

The time constant, $\frac{1}{\xi\omega_n}$, becomes:

$$\frac{1}{\xi\omega_n} = 2T \qquad \text{(IV-113)}$$

The actual angular velocity of the oscillation is given by

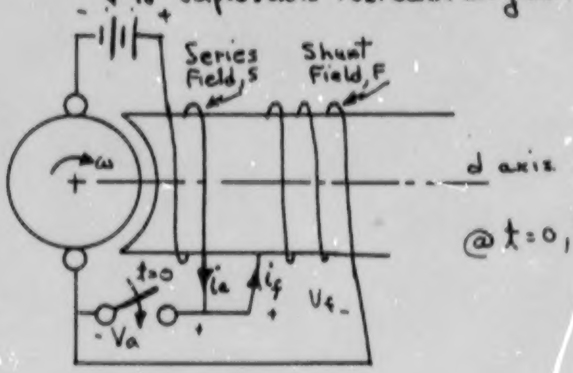
$$\omega_n = \sqrt{1 - \xi^2} \tag{IV-114}$$

An analysis of this type, while not yielding an exact solution, does give insight into dynamic behavior.

IV.6 SELF EXCITED D.C. MACHINE ON SHORT CIRCUIT: The 'ideal' d.c. machine would, if self excited, have a steady state short circuit current equal to zero because the field winding would also be short circuited. In reality, a steady state short current does persist because the residual magnetism does provide some excitation in the direct axis. If the machine were tested by driving it at the regular speed, with no field current and with the armature open circuited, a small voltage would manifest itself in the q axis brushes. This voltage, V_o, is the rotational voltage corresponding to residual magnetism. The effect of residual magnetism is shown in Figure IV-20. If the machine is driven at a constant speed, ω, and the armature circuit does not have connected load prior to the short circuit the armature current flowing is only that which flows in the shunt field. This is very small relative to rated current and especially small compared to the short circuit current. It will be neglected.

The circuit configuration we will investigate is shown in Figure IV-19. Most d.c. generators which are compound wound, i.e., both series and shunt field, are cumulative connected. The series and shunt field ampere turns act in the same direction.

& V_o – represents residual magnetism effect.



@ t=0, the switch is closed

Figure IV-19. Compound DC Machine on Short Circuit

The applicable equations from (III-99) through (III-106) are: (adding the S winding and taking into account residual magnetism).

$$V_f(s) = r_f i_f + L_f \frac{di_f}{dt} - L_{df} \frac{di_a}{dt} \tag{IV-115}$$

$$V_a(s) = - r_a i_a - L_a \frac{di_a}{dt} + \omega L_{df} i_f + k\ \omega L_{df} i_a + V_o \tag{IV-116}$$

The mutual inductance between the series field and the shunt field is L_{df}. There is a transformer type voltage in winding F due to changing current I_a. Also, there is a similar type voltage in the armature circuit due to changing current in the field circuit. This voltage is neglected because it has a small magnitude relative to other voltages in the armature circuit. The ratio of series field turns to shunt field turns is denoted by k. There are thus k times as many flux linkages resulting from I_a as from i_f.

For $t < 0$,

$$V_a(0) = V_f(0) = \omega L_{df} i_f(0) + V_o \tag{IV-117}$$

$$\therefore \quad i_f(0) = \frac{V_a(0) - V_o}{\omega L_{df}} \; ; \quad i_a(0) = 0 \tag{IV-118}$$

For $t > 0$

$$V_a = V_f = 0 \tag{IV-119}$$

Transforming (IV-115) and (IV-116), we have:

$$0 = r_f\left(1 + \frac{L_f}{r_f} s\right) I_f - L_f i_f(0) - L_{df} s\ I_a \tag{IV-120}$$

$$0 = - (r_a - k\ \omega L_{df})\left[1 + \frac{L_a}{r_a - k\ \omega L_{df}} s\right] I_a + \omega L_{df} I_f + \frac{V_o}{s} \tag{IV-121}$$

where $i_f(0)$ is given by (IV-118).

We will define the following time constants and apparent armature circuit resistance R_a.

$$R_a = r_a - k\ \omega L_{df} \tag{IV-122}$$

$$T_a = \frac{L_a}{R_a} \qquad \text{(IV-123)}$$

$$T_f = \frac{L_f}{r_f} \qquad \text{(IV-124)}$$

(IV-120) and (IV-121) can be solved for I_a as follows:

$$I_a = \frac{\begin{vmatrix} r_f(1 + T_f s) & L_f i_f(0) \\ \omega L_{df} & -\frac{V_o}{s} \end{vmatrix}}{\begin{vmatrix} r_f(1 + T_f s) & -L_d s \\ \omega L_{df} & -R_a(1 + T_a s) \end{vmatrix}} \qquad \text{(IV-125)}$$

$$I_a = \frac{r_f(1 + T_f s)\frac{V_o}{s} + \omega L_{df} \left[\frac{V_a(0) - V_o}{\omega L_{df}}\right]}{r_f R_a(1 + T_f s)(1 + T_a s) - \omega L_{df}^2 s} \qquad \text{(IV-126)}$$

$$= \frac{\frac{V_o}{R_a s}(1 + T_f s) + \frac{T_f}{R_a}[V_a(0) - V_o]}{1 + (T_f + T_a - \frac{\omega L_{df}^2}{R_a r_f})s + T_a T_f s^2} \qquad \text{(IV-127)}$$

In order to simplify our analysis, we will define new time constants, T_a' and T_f' such that

$$(1 + T_a' s)(1 + T_f' s) = 1 + (T_f + T_a - \frac{\omega L_{df}^2}{R_a r_f})s + T_a T_f s^2 \qquad \text{(IV-128)}$$

From (IV-128)

$$T_a' T_f' = T_a T_f \qquad \text{(IV-129)}$$

and:

$$T_a' + T_f' = T_a + T_f - \frac{\omega L_{df}^2}{R_a r_f} \qquad \text{(IV-130)}$$

In general, field circuit time constants are much larger than armature circuit time constants. Therefore, we can write:

$$T_f' \gg T_a' \; ; \quad T_f \gg T_a \tag{IV-131}$$

and (IV-130) becomes, with (IV-131):

$$T_f' \approx T_f - \frac{\omega L_{df}^2}{r_f R_a} \approx T_f\left(1 - \frac{\omega L_{df}^2}{R_a L_f}\right) \tag{IV-132}$$

and, substituting (IV-132) into (IV-129)

$$T_a' = T_a \frac{T_f}{T_f'} = \frac{T_a}{1 - \frac{\omega L_{df}^2}{R_a L_f}} \tag{IV-133}$$

Using (IV-128) in (IV-127) yields:

$$I_a = \frac{V_o}{R_a} \frac{(1 + T_f s)}{s(1 + T_f's)(1 + T_a's)} + [V_a(0) - V_o] \frac{T_f}{R_a} \frac{1}{(1 + T_f's)(1 + T_a's)} \tag{IV-134}$$

The inverse transform of (IV-134) gives $i_a(t)$ as:

$$i_a(t) = \frac{V_o}{R_a} \left(1 + \frac{T_a' - T_f}{T_f' - T_a'} \epsilon^{-t/T_a'} - \frac{T_f' - T_f}{T_f' - T_a'} \epsilon^{-t/T_f'}\right) +$$

$$+ \frac{T_f}{R_a} [V_a(0) - V_o]\left(\frac{1}{T_a' - T_f'}\right) \left(\epsilon^{-t/T_a'} - \epsilon^{-t/T_f'}\right) \tag{IV-135}$$

With the assumption that $T_f' \gg T_a'$, an approximate expression for (IV-135) is:

$$i_a(t) \approx \frac{V_o}{R_a} + \left[\frac{T_f V_o}{T_f' R_a} - \frac{V_o}{R_a} + \frac{T_f}{R_a T_f'} (V_a(0) - V_o)\right] \epsilon^{-t/T_f'} +$$

$$- \left[\frac{V_o T_f}{R_a T_f'} + \frac{T_f}{R T_f'} (V_a(0) - V_o)\right] \epsilon^{-t/T_a'} \tag{IV-136}$$

We will now define:

$$\frac{T_f}{T_f' R_a} = \frac{1}{R_a'} \tag{IV-137}$$

When this is substituted in (IV-136), we have;

$$i_a(t) \approx \frac{V_o}{R_a} + \left(\frac{V_a(0)}{R_a'} - \frac{V_o}{R_a}\right) \epsilon^{-t/T_f'} - \frac{V_a(0)}{R_a'} \epsilon^{-t/T_a'} \tag{IV-138}$$

Equation (IV-138) is plotted in Figure IV-20.

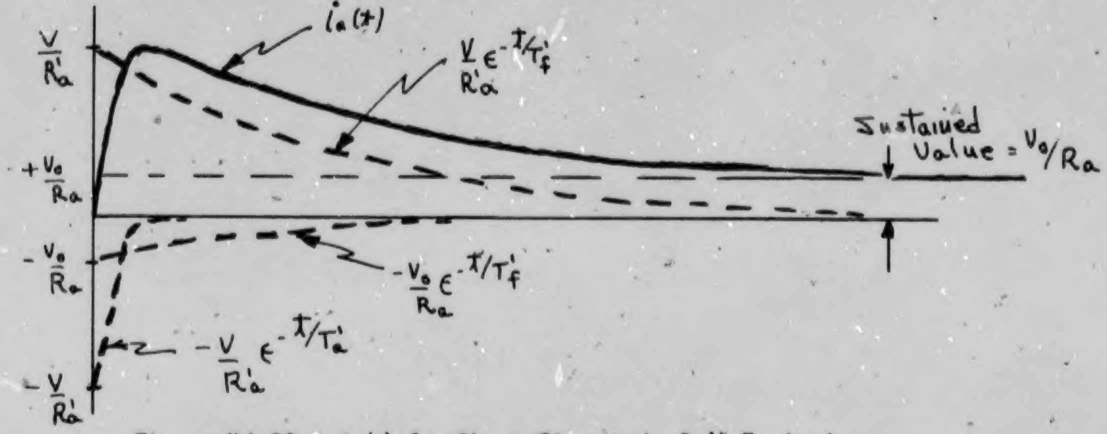


Figure IV-20. $i_a(t)$ for Short Circuited, Self Excited Compound DC Generator

Note the current rises to a value very nearly equal to:

$$\frac{V_a(0)}{R_a'} \tag{IV-139}$$

For this reason, R_a' is denoted as the "transient apparent armature resistance". The current finally settles down to a steady state value as determined by the voltage corresponding to residual magnetism and the apparent armature resistance. From (IV-122) this value is less than the actual armature circuit resistance.

Analysis based on valid assumptions, as accomplished here, are quite common in the study of the dynamic behavior of electric machines. Because of hysteresis effects and the impossibility of exact repeatability of experimental results extreme accuracy of solutions to machine problems is not warranted - nor is it meaningful. We will do the analysis of other types of machines by similar simplifying assumptions in later chapters.

CHAPTER V - CROSS FIELD MACHINES

The 2 Axis or Cross Field D.C. Machine (Metadyne)

V.1 INTRODUCTION: Machines with power windings and brushes in both the q and d axis are called metadynes. They can be used in a variety of applications - usually control systems - where power amplification with high speed response is necessary. Our analysis will deal with the basic metadyne operating as a power amplifier, i.e., we will introduce a signal into the field circuit and will extract an output with considerably greater power level from one of the armature circuits. The actual high power level output comes, of course, from conversion of mechanical power (to electric form) which comes into the machine through the shaft. The input signal controls, or modulates, the flow of mechanical power. The basic configuration which will be studied is shown in Figure V-1.

V.2 THE BASIC METADYNE ANALYSIS: In the basic configuration, current in coil F establishes flux in the d axis. A rotational voltage is induced in the Q winding, which is short circuited. Only a relatively low voltage in the Q winding is necessary for a heavy current i_q to flow. The current i_q results in an mmf and resulting flux in the q axis. The q axis flux linkages with the D winding result in a rotational voltage appearing at the brushes in the D winding. The D winding supplies the load.

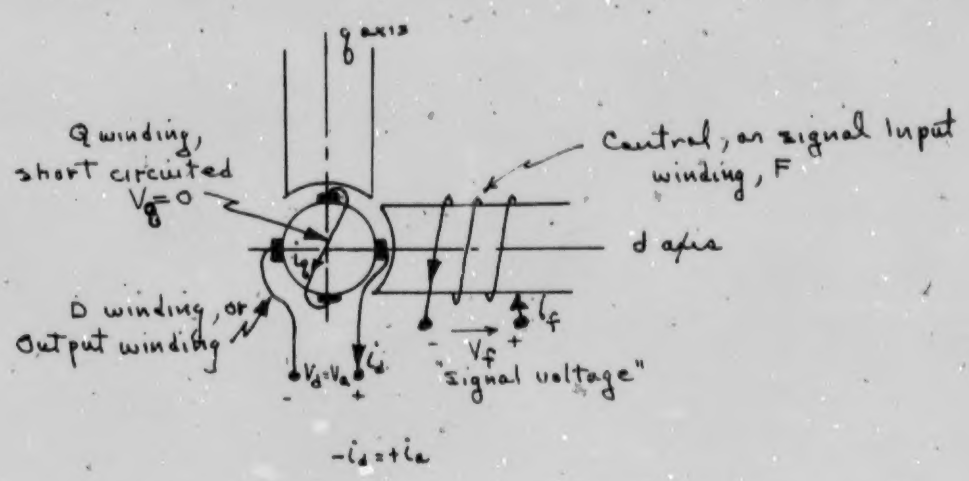

q axis
Q winding, short circuited $V_q = 0$
Control, or signal input winding, F
d axis
i_q
D winding, or Output winding
$V_d = V_a$
i_d
- +
i_f
- V_f +
"signal voltage"
$-i_d = +i_a$

Figure V-1. Basic Metadyne Configuration

The machine is driven at constant speed. The input signal is V_f, i_f. The machine does not have a G winding. The Q winding is short circuited. Therefore $V_q = 0$. Since power is taken from the D winding, $i_a = -i_d$. However, $V_d = V_a$ and $r_d = r_a$.

From (III-99) through (III-108), the describing equations are:

$$V_f = r_f i_f + L_f \frac{di_f}{dt} + L_{df}\frac{di_a}{dt} \tag{V-1}$$

$$V_a = -r_a i_a - L_a \frac{di_a}{dt} - L_{df}\frac{di_f}{dt} + \omega L_q i_q \tag{V-2}$$

$$0 = r_q i_q + L_q \frac{di_q}{dt} + \omega L_{df} i_f + \omega L_{df} i_a \tag{V-3}$$

Since a load R_L, L_L is connected across V_a,

$$V_a = R_L i_a + L_L \frac{di_a}{dt} \tag{V-4}$$

Assume the system is initially unexcited, i.e.,

$$i_f = 0,\ i_a = 0,\ V_a = 0 \tag{V-5}$$

The Laplace transform of (V-1, -2, -3, and -4) is:

$$V_f(s) = r_f(1 + T_f s)\, I_f(s) + L_{df} s\, I_a(s) \tag{V-6}$$

$$V_a(s) = R_L(1 + T_L s)\, I_a(s) \tag{V-7}$$

$$V_a(s) = -r_a(1 + T_a s)\, I_a(s) - L_{df} s\, I_f + \omega L_q I_q(s) \tag{V-8}$$

$$0 = r_q(1 + T_q s)\, I_q(s) + \omega L_{df} I_f(s) + \omega L_d I_a(s) \tag{V-9}$$

where:

$$T_f = \frac{L_f}{r_f};\ T_L = \frac{L_L}{R_L};\ T_a = \frac{L_a}{r_a};\ T_q = \frac{L_q}{r_q} \tag{V-10}$$

Also, since ω is a constant, it is customary to define:

$$\omega L_q = K_q;\ \omega L_d = K_d \approx \omega L_{df}$$

In order to gain insight into cross field machine behavior and to simplify the analysis, assumptions which are compatible with experimental results obtainable will be made. These assumptions are:

1. The transformer voltages induced in one winding by current change in another winding are negligible.
2. The time constants associated with the D and Q windings are nearly zero, or at least negligible.
3. The direct axis armature winding resistance is very much less than the load resistance.

With these assumptions, equations (V-6, -7, -8, -9) become:

$$V_f(s) = r_f(1 + \frac{L_f}{r_f} s)\ I_f(s) \qquad \text{(V-11)}$$

$$V_a(s) = R_L(1 + \frac{L_L}{R_L} s)\ I_a(s) \qquad \text{(V-12)}$$

$$V_a(s) = -\ r_a\ I_a(s) + K_q\ i_q(s) \qquad \text{(V-13)}$$

$$0 = r_q I_q(s) + K_d\ I_f(s) + K_d\ I_a(s) \qquad \text{(V-14)}$$

A block diagram representation of these equations is shown in Figure V-2. For convenience, assume a negative $V_f(s)$ in the input.

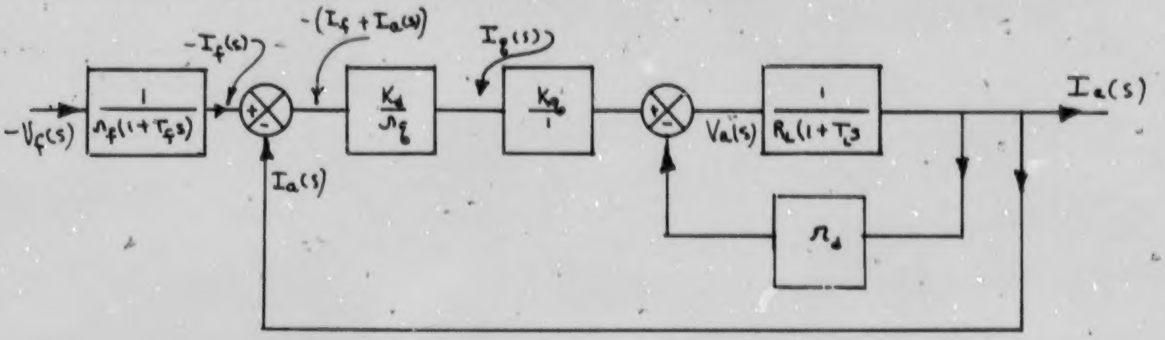


Figure V-2. Block Diagram for Basic Metadyne

We will eliminate $V_a(s)$ from (V-11, -12, -13, -14) leaving:

$$r_f(1 + T_f s)\; I_f(s) = V_f(s) \qquad \text{(V-15)}$$

$$-\,(r_a + R_L)\left(1 + \frac{L_L}{R_L + r_a}\, s\right) I_a(s) + K_q\; I_q(s) = 0 \qquad \text{(V-16)}$$

$$K_d I_f(s) + K_d I_a(s) + r_q I_q(s) = 0 \qquad \text{(V-17)}$$

Since:

$$R_L + r_a \approx R_L \qquad \text{(V-18)}$$

$$\frac{L_L}{R_L + r_a} \approx T_L \qquad \text{(V-19)}$$

We can solve for $I_a(s)$ as:

$$I_a(s) = \frac{\begin{vmatrix} r_f(1 + T_f s) & V_f(s) & 0 \\ 0 & 0 & K_q \\ K_d & 0 & r_q \end{vmatrix}}{\begin{vmatrix} r_f(1 + T_f s) & 0 & 0 \\ 0 & -(r_a + R_L)[1 + T_L s] & K_q \\ K_d & K_d & r_q \end{vmatrix}} \qquad \text{(V-20)}$$

From which:

$$I_a(s) = \frac{K_d K_q\; V_f(s)}{-r_f\; R_L\; r_q(1 + T_L s)(1 + T_f s) - K_d\; K_q\; r_f(1 + T_f s)} \qquad \text{(V-21)}$$

$$I_a(s) = \frac{K_d\; K_q\; V_f(s)}{r_f(R_L\; r_q + K_d\; K_q) + (r_f\; r_q\; L_L + r_q\; R_L L_f + K_d K_q L_f)s + r_q L_f L_L s^2} \qquad \text{(V-22)}$$

The minus sign means the current flow (and terminal voltage) are opposite to what we assumed they would be.

If we assume V_f is a step input, i.e., $V_f(s) = \frac{V_f}{s}$ and apply the Final Value Theorem, we have:

$$\lim_{t \to \infty} i_a(t) = \lim_{s \to 0} s\, I_a(s) = \frac{K_d\, K_q\, V_f}{r_f\,(R_L r_q + K_d K_q)} \qquad \text{(V-23)}$$

Refer back to the Block diagram for the metadyne, Figure V-2. Note the negative feedback loop which acts against the field (F winding) excitation. In effect, the load current tends to demagnetize the magnetic circuit. If we put another winding on the stator in the d axis and pass armature current through it, i.e., connect it in series with the D winding, and we connect it with a polarity such that it opposes this tendency to demagnetize (act against the F winding mmf) referred to above we can, by choosing the proper number of ampere turns, cancel out the $K_d\ i_a(s)$ term in the last expression in (V-14). This eliminates one of the negative feedback loops and is the principle of the "Amplidyne" or "compensated" metadyne. For such a configuration, (V-20) becomes:

$$I_a(s) = \frac{\begin{vmatrix} r_f(1 + T_f s) & V_f(s) & 0 \\ 0 & 0 & K_q \\ K_d & 0 & r_q \end{vmatrix}}{\begin{vmatrix} r_f(1 + T_f s) & 0 & 0 \\ 0 & -R_L(1 + T_L s) & K_q \\ K_d & 0 & r_q \end{vmatrix}} \qquad \text{(V-24)}$$

$$I_a(s) = \frac{K_d\ K_q\ V_f(s)}{-r_q r_f R_L(1 + T_f s)(1 + T_L s)} \qquad \text{(V-25)}$$

If we apply the Final Value Theorem to (V-25), assuming a step input V_f, we have:

$$\lim_{t \to \infty} i_a(t) = \frac{-\,K_d\ K_q\ V_f}{r_q\ R_L\ r_f} \qquad \text{(V-26)}$$

Comparison of (V-26) with (V-23) indicates a much higher "gain" (ratio of output to input) which is important if the machine is considered as an amplifier. To see why this is so, rewrite, for the block diagrams study, (V-11, -12, -13, -14) with the $K_d\ I_a(s)$ term not present and we have:

$$V_f \left[\frac{1}{r_f(1 + T_f s)}\right] = I_f(s) \tag{V-27}$$

$$I_a(s) = \frac{1}{R_L(1 + T_L s)}\ V_a(s) \tag{V-28}$$

$$K_q\ I_q(s) - r_a\ I_a(s) = V_a(s) \tag{V-29}$$

$$-\ I_f(s)\left(\frac{K_d}{r_q}\right) = I_q(s) \tag{V-30}$$

The Block diagram representation of (V-27, -28, -29, -30) is shown in Figure V-3. Note that we have eliminated one of the negative feedback loops. This explains why the "gain" is higher. Load current no longer acts in opposition to the input signal V_f.

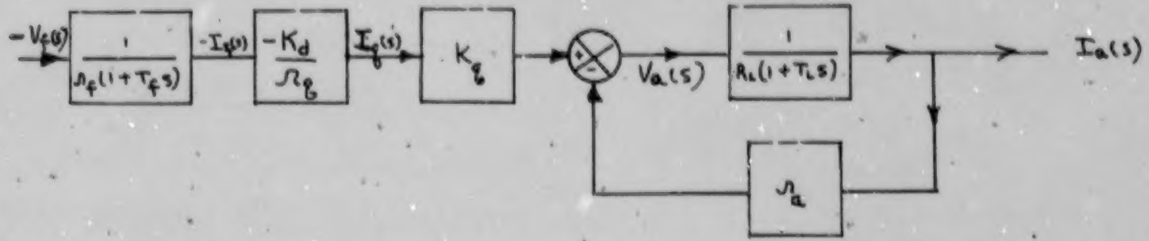


Figure V-3. Block Diagram for Amplidyne (Compensated Metadyne)

We can now calculate the power gain of the Amplidyne. This is of interest because the main usage of the Amplidyne is as a rotating amplifier.

Let

P_i = power into the field circuit

P_L = power output to the load

$$P_i = \frac{V_f^2}{r_f} \tag{V-31}$$

$$P_L = i_a^2\ R_L \tag{V-32}$$

$$\frac{P_L}{P_i} = \frac{i_a^{\,2} R_L r_f}{V_f^{\,2}} \qquad \text{(V-33)}$$

From (V-26):

$$i_a = \frac{K_d K_q V_f}{r_q R_L r_f} \qquad \text{(V-34)}$$

and

$$\frac{P_L}{P_i} = \frac{K_d^{\,2} K_q^{\,2}}{r_q^{\,2} R_L r_f} \qquad \text{(V-35)}$$

This can be an extremely high value in properly designed machine. Values on the order of 2000-50000 are not uncommon. Thus it is possible to get power on the order of kilowatts out of such a device. The amplidyne has been used as the link between low power error signals and the power stage in many practical servomechanisms. Usually an Amplidyne has multiple input windings, $F_1, F_2, \ldots$, etc. in order to provide for maximum flexibility.

The values of the various constants, for a specific machine, necessary for performance calculations can be determined by a rather simple input-output test using sinusoidal input and the technique of plotting the voltage gain as a function of input frequency (Bode plot). This test is described in the laboratory manual.

The following example will illustrate the effect of compensation on a metadyne and is illustrative of the power amplification possible with a well designed machine.

Example

Assume a "cross field" type of machine with the following rating and parameters:

2000 watts, 200 volts, 1800 rpm

r_f = 20 ohms; L_f = 4 henry

r_q = 0.4 ohms; L_q = 0.04 ohms

r_d = r_a = 0.4 ohms; L_d = L_a = 0.04 ohms

K_d = 120 volts per ampere

K_q = 240 volts per ampere

A series load of 20 ohms resistance and 2 henrys inductance is connected to the d-axis brushes.

Part 1. For a metadyne configuration (no compensating field) find:

a) V_f necessary to produce 200 volts d.c. steady state across the load.

b) Determine the steady state power gain.

200 volts dc steady state across the load of 20 ohms is an armature current of 10 amperes. From (V-23)

$$V_f = r_f \left(\frac{R_L\, r_q + K_d\, K_q}{K_d K_q} \right) I_a \qquad \text{(V-36)}$$

$$= 20 \left(\frac{20 \times .4 + 120 \times 240}{120 \times 240} \right) 10 \approx 200$$

In fact, in the steady state,

$$I_a = \frac{V_f}{r_f} \left[\frac{K_d\, K_q}{r_q\, R_L + K_d\, K_q} \right] = I_f \left[\frac{1}{\frac{r_q R_L}{K_d\, K_q} + 1} \right] \qquad \text{(V-37)}$$

In this example, for:

$$\frac{0.4\, R_L}{(120)(240)} \ll 0.1 \qquad \text{(V-38)}$$

or :

$$R_L \ll 7200 \text{ ohms}$$

$I_a \approx I_f$, independent of R_L and the metadyne acts as a current transformer wherein output current is equal to input current. The steady state power gain is

$$\frac{P_L}{P_i} = \frac{I_a^{\,2}\, R_L}{I_f^{\,2}\, r_f} = \frac{R_L}{r_f} \left[\frac{1}{\frac{r_q\, R_L}{K_d\, K_q} + 1} \right]^2 \qquad \text{(V-39)}$$

$$\approx \frac{R_L}{r_f} = \frac{20}{20} = 1$$

Part 2. Part 1 will be repeated for a metadyne with compensating field (an Amplidyne). In addition, find $V_a(t)$ for a step input of V_f necessary for I_a = 10 amperes in the steady state.

From (V-26)

$$V_f = \frac{r_q R_L r_f I_a}{K_d K_q} \tag{V-40}$$

$$= \frac{(0.4)(20)(20)(10)}{(120)(240)} = \frac{1}{18} \text{ volts}$$

$$\frac{P_L}{P_i} = \frac{I_a^2 R_L r_f}{V_f^2} = \frac{(10)^2(20)(20)}{(\frac{1}{18})^2} = 13 \times 10^6 \tag{V-41}$$

From (V-25)

$$I_a(s) = - \frac{(120)(240)(\frac{1}{18}) \frac{1}{s}}{(0.4)(20)(20)[1 + \frac{20}{4} s][1 + \frac{20}{2} s]} \tag{V-42}$$

$$V_a(s) = R_L(1 + T_L s) I_a(s) \tag{V-43}$$

$$V_a(s) = \frac{200 (1 + 10 s)}{s(1+5s)(1+10s)} = \frac{200}{s(1 + 5s)}$$

$$V_a(t) = 200(1 - \epsilon^{-t/5}) \text{ volts} \tag{V-44}$$

It should be noted that the "constant current" characteristic of the metadyne does not apply to the Amplidyne. In fact, in the steady state, for constant R_L, we see from (V-40) that

$$V_a \propto V_f \tag{V-45}$$

and the machine behaves as a voltage amplifier. From (V-40)

$$\frac{V_L}{V_f} = \frac{K_d K_q}{r_q r_f} = \frac{(120)(240)}{(0.4)(20)} = 3600$$

The advantages of a proportional amplifier capable of handling large amounts of power should be obvious!

CHAPTER VI - SYNCHRONOUS MACHINES

VI.1 THE REVOLVING MAGNETIC FIELD: In the polyphase synchronous machine, a number of coils are distributed around the periphery of one member of the machine (usually the stator) and are connected in such a fashion as to form a winding consisting of the appropriate number of phase windings. (i.e., three for a machine designed for a three phase system) for each pair of poles desired. The number of poles is determined from the specification of desired speed the machine is to either run at or to be driven at - depending on whether it is a motor or a generator (usually called alternator) and the frequency of the alternating current associated with the machine. As developed previously, angular velocity, frequency, and the number of poles are related by

$$f = \frac{n\,p}{120} = \frac{p\omega_m}{4} \qquad \text{(VI-1)}$$

where

f = frequency, hertz
p = number of poles
ω_m = angular velocity, rad/sec (mechanical)
n = angular velocity, rev/min

The phase windings, through distributed around the periphery (for full utilization of the magnetic structure) are located so that the axis of the various phase windings are displaced in space by an angle corresponding to the "time" angle associated with the electrical supply, i.e., 120° in space for three phase, 90° in space for two phase supply, etc. Figure (VI-1) indicates, schematically, the configuration for a two phase and a three phase winding in a 2 pole machine. The phase windings are shown concentrated for simplicity.

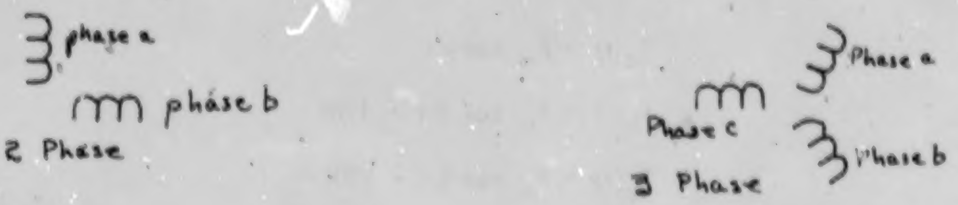


Figure VI-1. Schematic Diagram Indicating Space Displacement of Polyphase Windings

Each phase winding is supplied with alternating, sinusoidal current in the case of a motor, or each winding has such a current flowing in it in normal balanced operation of an alternator. These currents are displaced in time corresponding to 90° for a two phase system or 120° for a three phase system. An mmf results for each phase winding. The magnitude of each phase mmf pulsates with time and acts along the axis of that particular phase winding. The coil distribution and the design of the magnetic circuit are such as to secure as nearly as possible, a sinusoidal distribution in space of each of the phase mmfs. The phase mmfs combine to form a net, or resultant, mmf in the air gap of the machine. We can analyze the net mmf by examining the individual phase mmfs and (applying the principle of superposition) taking their sum. We will do this for a three phase machine.

Assume a symmetrical winding and voltage source, i.e., balanced. The instantaneous three phase currents are:

$$i_a = I_m \cos \omega t \qquad \text{(VI-2)}$$

$$i_b = I_m \cos (\omega t \mp 120^\circ) \qquad \text{(VI-3)}$$

$$i_c = I_m \cos (\omega t \pm 120^\circ) \qquad \text{(VI-4)}$$

where I_m is the maximum value and the time origin is arbitrarily taken as the instant when i_a is a positive maximum. The plus or minus is used so as to include either a-b-c or a-c-b phase sequence. We will denote space angle as θ and the phase mmfs, denoted by F, can be expressed at any instant by:

$$F_a = F_a(t) \cos \theta \qquad \text{(VI-5)}$$

$$F_b = F_b(t) \cos (\theta - 120) \qquad \text{(VI-6)}$$

$$F_c = F_c(t) \cos (\theta + 120) \qquad \text{(VI-7)}$$

where F(t) is the value of the mmf along its axis at any time t. Now F(t) is directly proportional to phase current (assuming a linear magnetic circuit) and we can utilize the time relationships in (VI-2, -3, -4) to establish the F(t)s for the various phases. Thus:

$$F_a(t) = F_m \cos \omega t \qquad \text{(VI-8)}$$

$$F_b(t) = F_m \cos (\omega t \mp 120) \qquad \text{(VI-9)}$$

$$F_c(t) = F_m \cos (\omega t \pm 120) \qquad \text{(VI-10)}$$

Note that the maximum value of the mmf in each phase is the same because we have assumed that the machine is balanced - i.e. symmetrical - for each phase winding. (The same number of turns per coil, same number of coils, same space distribution, etc.). The net air gap mmf will be denoted by $F(\theta,t)$ since it is a function of both space and time.

$$F(\theta,t) = F_a + F_b + F_c \qquad \text{(VI-11)}$$

$$= F_m \cos \theta \cos \omega t \quad F_m \cos (\theta - 120) \cos (\omega t \mp 120) +$$

$$+ F_m \cos (\theta + 120) \cos (\omega t \pm 120) \qquad \text{(VI-12)}$$

$$= F_m [\cos\theta\cos\omega t + (\cos\theta\cos 120 + \sin\theta\sin 120)$$

$$(\cos\omega t\cos 120 \pm \sin\omega t\sin 120) +$$

$$(\cos\theta\cos 120 - \sin\theta\sin 120)(\cos\omega t\cos 120 \mp \sin\omega t\sin 120)] \qquad \text{(VI-13)}$$

$$= F_m [\cos\theta\cos\omega t + (\frac{1}{4}\cos\theta\cos\omega t \mp \frac{\sqrt{3}}{4}\cos\theta\sin\omega t +$$

$$- \frac{\sqrt{3}}{4}\sin\theta\cos\omega t \pm \frac{3}{4}\sin\theta\sin\omega t) +$$

$$(\frac{1}{4}\cos\theta\cos\omega t \pm \frac{\sqrt{3}}{4}\cos\theta\sin\omega t + \frac{\sqrt{3}}{4}\sin\theta\cos\omega t +$$

$$\pm \frac{3}{4}\sin\theta\sin\omega t)] \qquad \text{(VI-14)}$$

$$= F_m [\frac{3}{2}\cos\theta\cos\omega t \pm \frac{3}{2}\sin\theta\sin\omega t] \qquad \text{(VI-15)}$$

$$= \frac{3}{2} F_m \cos(\theta \mp \omega t) \qquad \text{(VI-16)}$$

From (VI-16) we determine the following:

1) $F(\theta,t)$ is sinusoidally distributed in space and is revolving with an angular velocity ω.

2) the maximum value of $F(\theta,t)$ is 3/2 times the maximum value of any one phase mmf. In fact, it can be shown that it is q/2 times the maximum value of any one phase mmf, where q is the number of phases.

3) the direction of which $F(\theta,t)$ revolves is determined by the sign of ωt which is established by the phase seuqnce of the phase currents. Thus, for a motor, if we reversed any two phase leads of the supply source, we would reverse the direction of rotation of $F(\theta,t)$ (and thus reverse the direction of rotation of the motor).

4) the angular velocity, ω, is in electrical radians per second and $\omega = 2\pi f$ where f is supply frequency. The mechanical angular velocity, ω_m is related to the electrical angular velocity by

$$\omega = \frac{p}{2}\omega_m .$$

Thus:

$$\frac{\omega_m p}{4\pi} = f \qquad \text{(VI-17)}$$

verifying our earlier derivations for this relationship.

It has also been shown previously that a necessary condition for the production of steady torque is that the magnetic field associated with the rotor and with the stator be stationary with respect to each other (not necessarily with respect to a stationary reference frame). Since the stator mmf (and resulting magnetic field) rotates at a velocity given by (VI-17), it is then obvious that the steady state rotor speed must also be equal to this value. (Recall that, for a synchronous machine, the rotor is excited by direct current and the rotor field is therefore stationary with respect to the rotor).

The 2 pole synchronous machine is depicted in Figure VI-2.

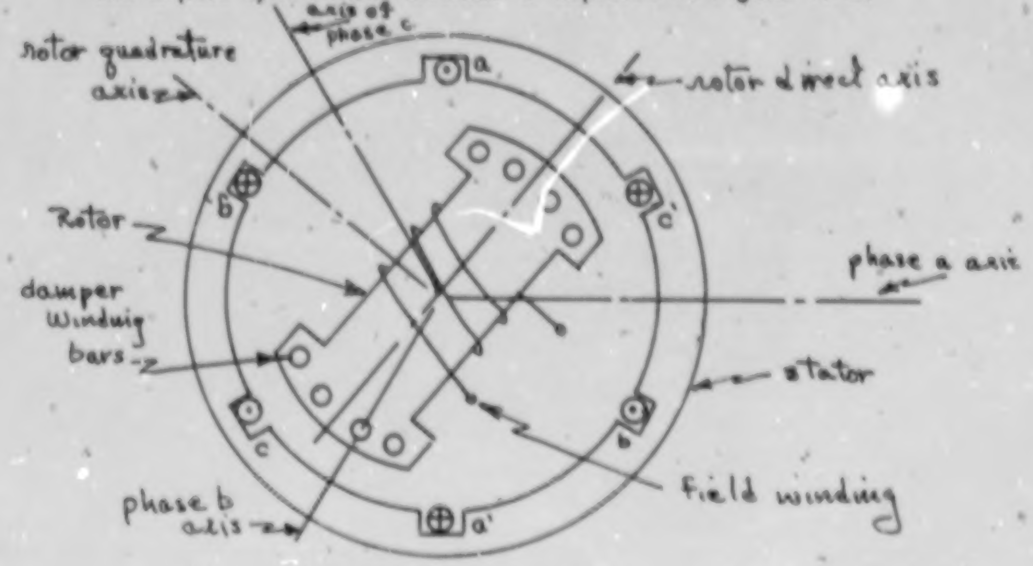


Figure VI-2. 2 Pole Synchronous Machine

Figure VI-2 depicts a stator with concentrated coils, a-a', b-b', c-c', forming the polyphase stator winding. It should be emphasized that this winding is for a three phase, 2 pole machine. The rotor has a coil wound on it and supplied with d.c. current via slip rings. In the general case, the rotor also has "damper" or amortissuer windings imbedded in the pole face. These windings are shorted together by connecting bars at the end of the rotor and thus form a "squirrel cage" type winding. In some machines, the connecting bars at the end extend from pole to pole and thus definite windings whose axis are centered along the rotor direct and quadrature axis.

Even where no deliberate pole to pole connection is made, the rotor iron itself forms such a circuit, allowing eddy currents to flow during transient or unbalanced conditions. In the steady state, since these damper windings rotate at the same speed as the stator and rotor fields (and thus do not have relative motion to magnetic fields) no induced voltage exists in them, no current flows in them, and they are completely inactive. However, if a transient situation develops whereby the rotor and the stator field are rotating at differnt velocities, induced voltages do appear, currents flow and "asynchronous" torques exist. These torques are analogous to these present in the induction motor which will be studied later. Indeed, if this was not possible, the synchronous motor could not be self-starting because the average torque, with the rotor stationary, would be zero. In practice, the motor starts as an induction motor, comes up to nearly synchronous speed, field excitation is applied, and the motor pulls into step and then runs as a synchronous motor at synchronous speed.

The electrical schematic of the synchronous machine, including the short circuited damper winding is shown in Figure IV-3. As an approximation, the currents in the damper winding, or in the paths formed by the iron, are assumed to flow in two closed circuits, one in each axis. The effective circuits are denoted as KD and KQ windings.

Note that the stator windings are coupled magnetically with each other and with the rotor windings. Also, since the rotor may be "salient" (as shown in Figure VI-2), rather than "cylindrical" (or magnetically symmetrical) the magnetic performance is higher in the direct axis then in the quadrature axis. Because of the difference in permeance, the magnetic coupling (and thus the various self and mutual inductances) varies with rotor position and time (because the rotor turns).

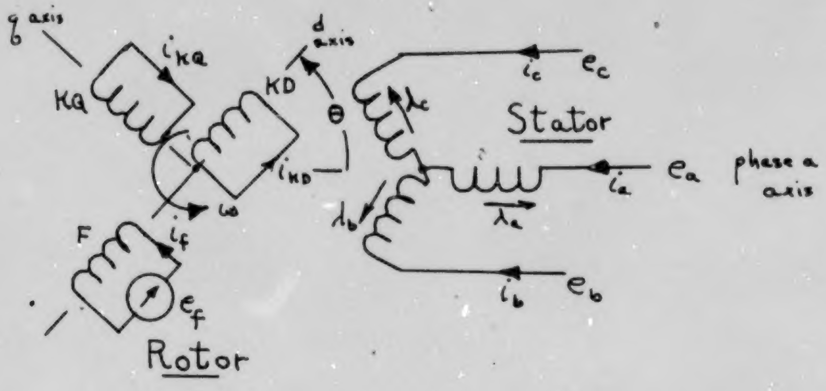


Figure VI-3. Rotor and Stator Circuits of 3 Phase Synchronous Motor, with Damper Circuits

Note that in the generalized and D.C. machine, the F winding was stationary and the D winding rotates. The direction, with assumed current convention, is D clockwise with respect to F. If D is stationary and F rotates, the same convention results in F rotating counter-clockwise with respect to D.

VI.2 SYNCHRONOUS MACHINE FLUX LINKAGES: The damper circuits referred to are shown as KD and KQ. The field winding, or excitation source, is designated as F. The current directions are taken to be for a motor. At any instant of time we can write expressions for the flux linkages. For example, the flux linkages, λ_a, in phase, "a", are:

$$\lambda_a = \mathcal{L}_{aa} i_a - \mathcal{L}_{ab} i_b - \mathcal{L}_{ac} i_c + \mathcal{L}_{af} i_f + \mathcal{L}_{akd} i_{kd} + \mathcal{L}_{akq} i_{kq} \qquad \text{(VI-18)}$$

where $\mathcal{L}_{aa}$ is the self inductance of phase "a".

$\mathcal{L}_{ab}, \mathcal{L}_{ac}, \mathcal{L}_{af}, \mathcal{L}_{akd}, \mathcal{L}_{akq}$ are mutual inductances of the various other windings with respect to phase "a" winding.

As the rotor changes position, the permeance of the magnetic circuit changes. Thus the various inductance values are functions of time. However, the variations are periodic and can be represented by Fourier Series expansion. Because of the distributed nature of the windings and the design objectives, it has been determined (from many tests) that we need only consider the Fundamental term of the Fourier series in order to adquately represent the angular variation of inductance. Thus, since the mutual inductance between a phase and the rotor circuits is a maximum when a rotor winding is aligned with the axis of a phase, we can write:

$$\mathcal{L}_{af} = L_{af} \cos\theta \qquad \text{(VI-19)}$$

$$\mathcal{L}_{akd} = L_{akd} \cos\theta \qquad \text{(VI-20)}$$

$$\mathcal{L}_{akq} = -L_{akq} \sin\theta \qquad \text{(VI-21)}$$

The reason for the minus sign should be obvious when one examines the configuration shown in Figure VI-3. Similar equations can be written for phases "b" and "c" with θ replaced by $(\theta - 120^\circ)$ and $(\theta - 240^\circ)$ respectively.

The stator self inductance, $\mathcal{L}_{aa}$, of phase "a" has a variation with angle θ as shown in Figure VI-4.

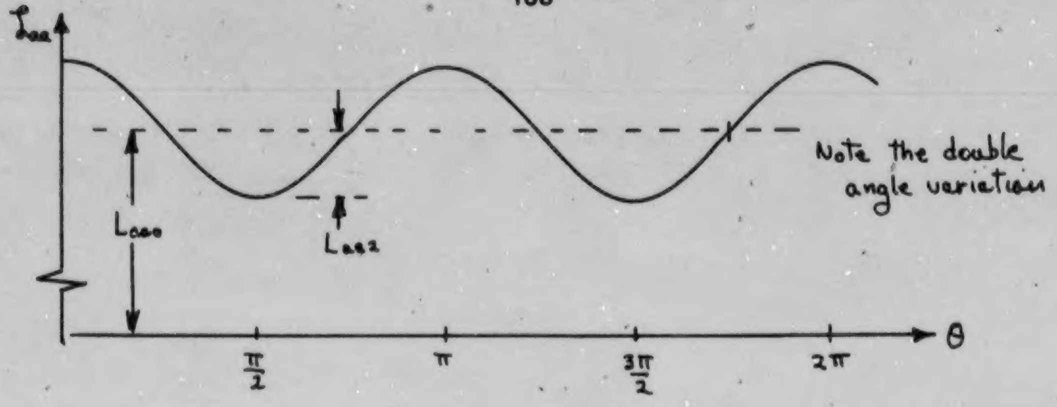


Figure VI-4. $\mathcal{L}_{aa}$ as a Function of θ

Thus,

$$\mathcal{L}_{aa} = L_{aao} + L_{aa2} \cos 2\theta \tag{VI-22}$$

The mutual inductance between "a" and "b" phases is a maximum when $\theta = +30^\circ$ or 210°, and between "a" and "c" phases when $\theta = -30^\circ$ or 150° and it has a double angle variation. We can, therefore, write:

$$\mathcal{L}_{ab} = L_{abo} + L_{ab2} \cos (2\theta - 60) \tag{VI-23}$$

$$\mathcal{L}_{ac} = L_{aco} + L_{ac2} \cos (2\theta + 60) \tag{VI-24}$$

The coefficients L_{abo} and L_{aco}, as well as L_{ab2} and L_{ac2} are respectively equal as a result of symmetry of the stator. Further, L_{aa2}, L_{ab2}, and L_{ac2} must be equal because it is variation in the air gap permeance which causes the variation and the variation is the same in each case. We can now write the flux linkages for three phase windings, using the angular variations, as follows:

$$\begin{aligned} \lambda_a = {} & i_a [L_{aao} + L_{aa2} \cos 2\theta] - L_{abo} (i_b + i_c) - L_{aa2} i_b \cos (2\theta + 60) + \\ & - L_{aa2} i_c \cos (2\theta - 60) + L_{af} i_f \cos \theta + L_{akd} i_{kd} \cos \theta + \\ & - L_{akq} i_{kq} \sin \theta \end{aligned} \tag{VI-25}$$

Similarly:

$$\lambda_b = i_b[L_{aao} + L_{aa2}\cos(2\theta + 120)] - L_{abo}(i_a + i_c) - L_{aa2}i_c\cos(2\theta + 180) +$$

$$- L_{aa2}i_a\cos(2\theta + 60) + (L_{af}i_f + L_{akd}i_{kd})\cos(\theta - 120) +$$

$$- L_{akq}i_{kq}\sin(\theta - 120) \qquad \text{(VI-26)}$$

and:

$$\lambda_c = i_c\,[L_{aao} + L_{aa2}\cos(2\theta - 120)] - L_{abo}(i_a + i_b) - L_{aa2}i_a\cos(2\theta - 60) +$$

$$- L_{aa2}i_b\cos(2\theta + 180) + (L_{af}i_f + L_{akd}i_{kd})\cos(\theta - 240) +$$

$$- L_{akq}i_{kq}\sin(\theta - 240) \qquad \text{(VI-27)}$$

The equations for flux linkates linking the field (F) winding and the damper windings (KD and KQ) are:

$$\lambda_f = L_f i_f + L_{fkd}i_{kd} + L_{af}[i_a\cos \quad + i_b\cos(\theta - 120) + i_c\cos(\theta - 240)] \qquad \text{(VI-28)}$$

$$\lambda_{kd} = +L_{fkd}i_f + L_{kd}i_{kd} + L_{akd}[i_a\cos\theta + i_b\cos(\theta - 120) + i_c\cos(\theta - 240)] \qquad \text{(VI-29)}$$

$$\lambda_{kq} = L_{kq}i_{kq} - L_{akq}[i_a\sin\theta + i_b\sin(\theta - 120) + i_c\sin(\theta\ 240)] \qquad \text{(VI-30)}$$

VI.3 THE a-b-c to d-q-o TRANSFORMATION: Recall that, in a previous section, we discussed the possibility (by suitable transformation) of an analysis of all types of machines on a "generalized" basis. The generalized machine has all coils, windings, mmfs, etc. acting along either the direct or quadrature axis of the rotor. λ_a, λ_b, λ_c are oriented along the axis of the phase windings and, since they are stationary and the rotor rotates they cannot, individually, be considered as acting along either the q or d axis of the rotor. However, in Section (VI.1) it was shown that the mmfs of the individal phases combine to form a resultant mmf. This mmf is responsible for a resultant air gap flux density and flux, and of course flux linkages). This mmf is revolving at synchronous speed, as is the rotor, and we should be able to use this information to determine a transformation that will relate λ_a, λ_b, and λ_c to components of the net revolving flux linkage that are along the d and q axis. Since mmf results from current flow, and currents are associated with voltage, the same transformation should also be valid for current and voltage. In fact, we will describe the transformed mmfs, F_d and F_q in terms of the instantaneous phase currents.

Refer to Figure VI-5. This illustrates how the "a" phase mmf has components along the d axis, θ degrees ahead of the "a" phase axis and along the q axis, (θ + 90°) ahead of the "a" phase axis.

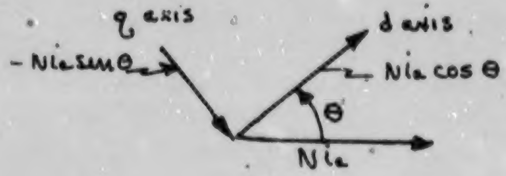


Figure VI-5. Phase "a" mmf Acting Along the d and q axis

The relationship for the three phases is shown in Figure VI-6.

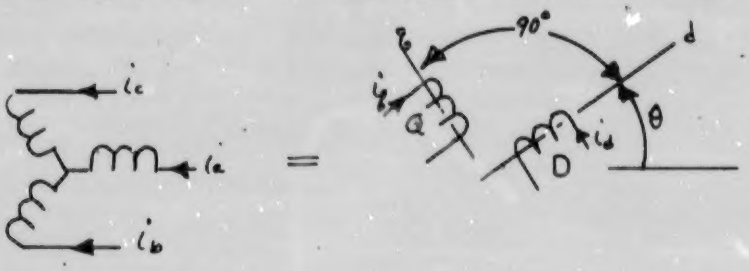


Figure VI-6. Phase d,q Relationships

Refer to Figure VI-5 and VI-6:

$$F_d = N[i_a \cos\theta + i_b \cos(\theta - 120) + i_c \cos(\theta + 120)] = k_d i_d \quad \text{(VI-31)}$$

$$F_q = N[-i_a \sin\theta - i_b \sin(\theta - 120) - i_c \sin(\theta + 120)] = k_q i_q \quad \text{(VI-32)}$$

where

F_d, F_q are the components of the net air gap mmf acting along the d,q, axis.

N is the effective number of turns of each phase winding.

k_d, k_q are the effective number of turns on fictitious windings, D, Q, located on the d,q, axis

i_d, i_q are currents that must flow in windings D, Q, and result in the same net air gap mmf as established by actual currents i_a, i_b, i_c

Note that 3 variables are present in the phase quantities, (i_a, i_b, and i_c) and we must have 3 variables in the transformed quantities in order to perform an inverse transformation. Now the two currents, i_d and i_q, as defined produce the correct magnetic field. Therefore, the third variable - we will denote it as i_o - must be defined in such a way as to produce no space fundamental mmf in the air gap. If no neutral, or 4th wire current, exists this requirement can be met if we define it as:

$$i_o = \frac{1}{3}(i_a + i_b + i_c) \qquad \text{(VI-33)}$$

The factor 1/3 is completely arbitrary but is used because this is the way in which "zero sequence" current is defined in the method of Symmetrical Components - used in the analysis of unbalanced 3 phase power systems. Since i_o produces no flux linking the rotor, it must be associated with the stator leakage inductances. For reasons which will be apparent later, the ratios N/k_d and K/k_q are taken as 2/3. Then, we can write (VI-31-32, and -33) as:

$$\begin{vmatrix} i_d \\ i_q \\ i_o \end{vmatrix} = \frac{2}{3} \begin{vmatrix} \cos\theta & \cos(\theta-120) & \cos(\theta+120) \\ -\sin\theta & -\sin(\theta-120) & -\sin(\theta+120) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix} \bullet \begin{vmatrix} i_a \\ i_b \\ i_c \end{vmatrix} \qquad \text{(VI-34)}$$

From the inverse of (VI-34), we have:

$$\begin{vmatrix} i_a \\ i_b \\ i_c \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin & 1 \\ \cos(\theta-120) & -\sin(\theta-120) & 1 \\ \cos(\theta+120) & -\sin(\theta+120) & 1 \end{vmatrix} \bullet \begin{vmatrix} i_d \\ i_q \\ i_o \end{vmatrix} \qquad \text{(VI-35)}$$

Since this transformation is valid for transforming stationary phase quantities to quantities along the d,q, rotor axis - which are stationary with respect to the rotor axis, the transformation should eliminate the time variation of the inductances and is equally valid for either flux linkages, voltages or currents, as argued previously.

Using the transformation defined above, we obtain from (VI-25, -26, -27)

$$\lambda_o = (L_{aao} - 2L_{abo})\, i_o \tag{VI-36}$$

$$\lambda_d = (L_{aao} + L_{abo} + \frac{3}{2} L_{aa2})\, i_d + L_{af} i_f + L_{akd}\, i_{kd} \tag{VI-37}$$

$$\lambda_q = (L_{aao} + L_{abo} - \frac{3}{2} L_{aa2})\, i_q + L_{akq}\, i_{kq} \tag{VI-38}$$

We can define new, fictitious, armature inductances as:

$$L_o = L_{aao} - 2L_{abo} \tag{VI-39}$$

$$L_d = L_{aao} + L_{abo} + \frac{3}{2} L_{aa2} \tag{VI-40}$$

$$L_q = L_{aao} + L_{abo} - \frac{3}{2} L_{aa2} \tag{VI-41}$$

L_d, L_q, L_o are defined inductances that relate to measurable inductances that actually exist. However, even though they are only defined, it happens that these defined inductances are quantities whose value can be obtained relatively easily from test data. We can also use the current transformations in (VI-35), to obtain rotor flux linkages:

$$\lambda_f,\ \lambda_{kd},\ \lambda_{kq}$$

In terms of:

$$i_d,\ i_q,\ i_f$$

Thus, from (VI-28,-29,and -30):

$$\lambda_f = L_f i_f + L_{fkd} i_{kd} + \frac{3}{2} L_{af} i_d \tag{VI-42}$$

$$\lambda_{kd} = L_{fkd} i_f + L_{kd} i_{kd} + \frac{3}{2} L_{akd}\, i_d \tag{VI-43}$$

$$\lambda_{kq} = L_{kq} i_{kq} + \frac{3}{2} L_{akq}\, i_q \tag{VI-44}$$

Also, substituting (VI-39, -40,-41) in VI-36,-37,-38) yields:

$$\lambda_o = L_o i_o \tag{VI-45}$$

$$\lambda_q = L_q i_q + L_{akq} i_{kq} \tag{VI-46}$$

$$\lambda_d = L_d i_d + L_{af} i_f + L_{akd} i_{kd} \tag{VI-47}$$

Now, the voltage equations for the actual stator and rotor circuits are:

$$V_a = \frac{d\lambda_a}{dt} + r\, i_a \tag{VI-48}$$

$$V_b = \frac{d\lambda_b}{dt} + r\, i_b \tag{VI-49}$$

$$V_c = \frac{d\lambda_c}{dt} + r\, i_c \tag{VI-50}$$

$$V_f = \frac{d\lambda_f}{dt} + r_f i_f \tag{VI-51}$$

$$0 = \frac{d\lambda_{kd}}{dt} + r_{kd} i_{kd} \tag{VI-52}$$

$$0 = \frac{d\lambda_{kq}}{dt} + r_{kq} i_{kq} \tag{VI-53}$$

where r is the stator winding resistance, r_{kd} and r_{kq} are equivalent kd and kq circuit resistances.

If we define V_d, V_q, and V_o with the same transformation used for current and flux linkages and transform the phase voltages and currents in (VI-48,-49,-50) to d-q-o quantities we have,

$$V_d = \frac{d\lambda_d}{dt} - \lambda_q \frac{d\theta}{dt} + r\, i_d \tag{VI-54}$$

$$V_q = \frac{d\lambda_q}{dt} + \lambda_d \frac{d\theta}{dt} + r\, i_q \tag{VI-55}$$

$$V_o = \frac{d\lambda_o}{dt} + r\, i_o \tag{VI-56}$$

We recognize that $\frac{d\theta}{dt} = \omega$ speed. Also, if we assume that a common mutual flux links all windings on one axis and that this common mutual flux differs from the total flux linking any one winding by the amount of leakage flux associated with that particular winding we can make additional simplifications in the equatiuons we have derived.

The equations as derived so far are in terms of amperes, volts, henries, etc., i.e., they are actual quantities. It is appropriate (prior to introducing the common mutual fluxes and the leakage fluxes) to convert to a per unit system in such a fashion as to make the flux linkage equations have reciprocal mutual inductances - that is to say the per unit value of the mutual inductance along a given axis is the same regardless of which winding we are considering it from.

VI.4 THREE PHASE MACHINE PER UNIT QUANTITIES: The basic considerations involving the use of the per unit system were developed in Section III.2 The development in that section dealt with a single phase machine, or a DC machine. The section will extend the per unit usage tp cover three phase machines. Two very important concepts developed in Section III.2 which also apply here as:

1. in coupled coils, or windings, the same voltampere base should be chosen in each coil (to make the mutual inductance the same when viewed from either coil)
2. base value of electrical angular velocity should be chosen as equal to 1.0 if the time relationships are to be preserved in the analysis.

For the three phase machine, base voltage and base current in the armature (power winding) are best taken as the maximum of the instantaneous phase values corresponding to machine voltampere rating. Thus; if

V_{mb} = base voltage which is also the maximum or crest value of phase voltage

I_{mb} = base current which is also the maximum, or crest value of the phase current

$$3\left(\frac{V_{mb}}{\sqrt{2}}\right)\left(\frac{I_{mb}}{\sqrt{2}}\right) = \text{3 phase machine voltampere rating} \qquad \text{(VI-57)}$$

$$= \text{3 phase voltampere base}$$

$$= \frac{3}{2} V_{mb} I_{mb} = VA_{base} \text{ 3 phase} \qquad \text{(VI-58)}$$

To see how this corresponds to the equations developed for the synchronous machine, note the flux linkages in the field winding due to current in the D winding and flux linkages in the D winding due to current in the field winding.

For the former; from (VI-42):

$$\lambda_f \text{ due to } i_d = \frac{3}{2} L_{af} i_d \tag{VI-59}$$

For the latter, from (VI-47):

$$\lambda_d \text{ due to } i_f = L_{af} i_f \tag{VI-60}$$

If these two quantities are to be equal, in per unit;

$$\frac{3}{2} \frac{L_{af}\, i_d}{L_{f\ base}\, i_{f\ base}} = \frac{L_{af}\, i_f}{L_{d\ base}\, i_{d\ base}} \tag{VI-61}$$

If $\omega_{base} = 1.0$; (VI-61) can be written as

$$\frac{3}{2} \frac{L_{af}\, i_d\, i_{f\ base}}{(Z_{f\ base}\, i_{f\ base})\, i_{f\ base}} = \frac{L_{af}\, i_f\, i_{d\ base}}{(Z_{d\ base}\, i_{d\ base})\, i_{d\ base}} \tag{VI-62}$$

$$\frac{3}{2} \frac{i_d / i_{d\ base}}{V_{f\ base}\, i_f} = \frac{i_f / i_{f\ base}}{V_{d\ base}\, i_{d\ base}} \tag{VI-63}$$

If

$$\frac{3}{2} V^{3}_{d\ base}\, i_{d\ base} = V_{f\ base}\, i_{f\ base} = VA_{base}\ 3 \text{ phase} \tag{VI-64}$$

The flux linkages are equal when:

$$\frac{i_d}{i_{d\ base}} = \frac{i_f}{i_{f\ base}} \tag{VI-65}$$

(VI-65) equates the per unit current in the two windings. Thus, the conclusion is that if the volt ampere base in the field and the armature are equal - and are in turn the three phase volt-ampere rating of the machine, then we have equal per unit flux linkages in the two circuits (due to mutual coupling) when the per unit currents in the armature and field are equal. Thus the mutual inductances are equal and reciprocal. The same argument can be extended to the mutual inductances between the damper circuits and the armature circuits, i.e., the volt ampere base in the damper circuits must be the same as the 3 phase volt ampere rating of the machine.

VI.5 SUMMARY OF SYNCHRONOUS MACHINE EQUATIONS: The synchronous machine three phase stator winding is transformed into D, Q, type windings by the d-q-o transformation (VI-34). The synchronous machine configuration is similar to that of the Generalized Machine except:

1. The D and Q windings are on the stator.
2. The "G" winding on the Generalized Machine is denoted as the KQ winding, is closed on itself and is on the rotor.
3. The F winding is on the rotor.
4. Another winding, denoted as the KD winding is coaxial with the F winding and is also on the rotor. It is closed on itself electrically.
5. A "zero" component winding exists mathematically in order to permit dqo to a-b-c (phase quantity) transformation. It does not contribute flux linkages to the rotor winding and is thus associated with stator leakage reactance.

Corresponding to (III-47 through -50) we can express the self, mutual and leakage inductances for the F, D, Q, KD, KQ windings and for the zero component winding. Similarly we can then write the expressions for flux linkages as developed in (VI-42 through VI-47). These correspond to (III-51, -52,-54,-55) for the Generalized Machine. Finally, from (VI-51 through -56) we have the voltage equations for various windings. These are similar to (III-99 through -102) for the Generalized Machine. The two equations describing the electromechanical dynamics are, of course, the same for both machines since power is converted from mechanical to electrical form, or vice versa, in the armature circuit only (the D,Q, windings).

The equations for the synchronous machine are repeated immediately below. Comparison of these with the corresponding equations for the Generalized Machine indicates that they could have been written by analogy.

L_{df} = mutual inductance, d axis windings

L_{qg} = mutual inductance, q axis windings

L_o = zero component inductance

L_f = self inductance of F winding

ℓ_{-} = mutual inductance of _ winding, etc.

$$L_f = L_{df} + \ell_f \qquad \text{(VI-66)}$$

$$L_d = L_{df} + \ell_d \qquad \text{(VI-67)}$$

$$\ell_{kd} = L_{df} + \ell_{kd} \tag{VI-68}$$

$$L_q = L_{qg} + \ell_q \tag{VI-69}$$

$$L_{kq} = L_{qg} + \ell_{kq} \tag{VI-70}$$

$$\lambda_f = L_f i_f + L_{df} i_{kd} + L_{df} i_d \tag{VI-71}$$

$$\lambda_d = L_{df} i_f + L_{df} i_{kd} + L_d i_d \tag{VI-72}$$

$$\lambda_{kd} = L_{df} i_f + L_{kd} i_{kd} + L_{df} i_d \tag{VI-73}$$

$$\lambda_q = L_q i_q + L_{qg} i_{kq} \tag{VI-74}$$

$$\lambda_{kq} = L_{qg} i_q + L_{kq} i_{kq} \tag{VI-75}$$

$$\lambda_o = L_o i_o \tag{VI-76}$$

$$V_f = r_f i_f + \frac{d\lambda_f}{dt} \tag{VI-77}$$

$$V_d = r_d i_d + \frac{d\lambda_d}{dt} - \omega\lambda_q \tag{VI-78}$$

$$0 = r_{kd} i_{kd} + \frac{d\lambda_{kd}}{dt} \tag{VI-79}$$

$$V_q = r_q i_q + \frac{d\lambda_q}{dt} + \omega\lambda_d \tag{VI-80}$$

$$0 = r_{kq} i_{kq} + \frac{d\lambda_{kq}}{dt} \tag{VI-81}$$

$$V_o = r_o i_o + \frac{d\lambda_o}{dt} \tag{VI-82}$$

$$T_e = k(i_q \lambda_d - i_d \lambda_q) \tag{VI-83}$$

$$T_e = T_L(\omega) + J\frac{d\omega}{dt} \tag{VI-84}$$

Observations which can be made at this time with respect to these equations are:

1. under balanced conditions, no zero currents and voltages exist.

2. Under steady state conditions, (rotor with same angular velocity as the stator magnetic field) i_{kd}, i_{kq} are zero, since only transformer voltages are present in these windings.

From a historical basis, (in honor of R. H. Park) the equations in d-q-o form are usually referred to as "Parks" equation. The analysis using these equations is also sometimes referred to as the "Two Reaction Method".

VI.6 EQUATION REARRANGEMENT FOR BLOCK DIAGRAM REPRESENTATION: From equations (VI-71 through VI-84) we note that we have 14 simultaneous equations which are non-linear if speed varies - which it may do in the dynamic or transient state. It should be emphasized that the average speed must be synchronous, i.e, the same as the speed of the stator field, if the motor is to remain in the synchronous motor mode. However, there can be oscillations about this speed. Some rearrangement of the equations is desirable in order to more easily represent the system of equations in block diagram form or for simulation in computation. In general, we are not concerned with knowing the actual value of the flux linkages. Therefore we will do some combining and rearranging to eliminate these from our block diagram. For a block diagram, we require transfer functions. The first such transfer function we will obtain is the easiest, i.e., for zero sequence quantities. From (VI-76 and -82) the equations involving zero sequence quantities are:

$$V_o = \frac{d\lambda_o}{dt} + r\, i_o \qquad \text{(VI-82)}$$

$$\lambda_o = L_o i_o \qquad \text{(VI-76)}$$

or:

$$V_o = L_o \frac{d\, i_o}{dt} + r\, i_o \qquad \text{(VI-85)}$$

Recall that the use of block diagram based on transfer function is valid only for zero initial conditions. Suppose we consider only incremental changes in the variable. Since they are changes that take place only after t = 0, the initial conditions for incremental changes must be zero and thus we can represent incremental changes by block diagrams representing transfer functions. This can be demonstrated by considering

(VI-85) and denoting incremental values of the variable by a Δ preceding the variable. The value of the variable at t=0 (the steady state value preceding the change) is denoted by the variable followed by (0). Thus:

$$(V_o(0) + \Delta V_o) = L_o \frac{d}{dt} (i_o(0) + \Delta i_o) + r\, i_o(0) + r\, \Delta i_o \qquad \text{(VI-86)}$$

and:

$$V_o(0) = r\, i_o(0) \qquad \text{(VI-87)}$$

Subtracting (VI-87) from (VI-86) yields:

$$\Delta V_o = L_o \frac{d}{dt} \Delta i_o + r\, \Delta i_o \qquad \text{(VI-88)}$$

from which:

$$\Delta i_o = \Delta V_o \left[\frac{1}{r(1 + \frac{L_o}{r} s)} \right. \qquad \text{(VI-89)}$$

After taking the Laplace transform. We can represent this in block diagram form as:

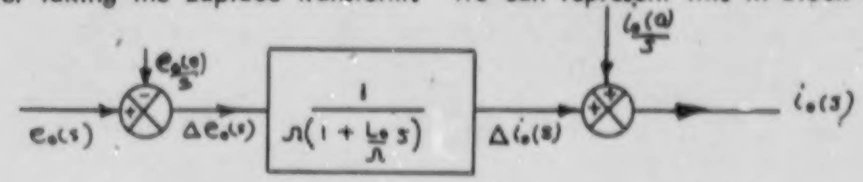


Figure VI-7. Zero Sequence Block Diagram

That this yields correct results can be seen from taking a step input

$$V_o(s) = \frac{\Delta V_o}{s} \qquad \text{(VI-90)}$$

Now

$$V_o(s) = \frac{\Delta V}{s} + \frac{V_o(0)}{s} \qquad \text{(VI-91)}$$

$$i_o(s) = [V_o(s) - \frac{V_o(0)}{s}][\frac{1}{(r + L_o s)}] + \frac{i_o(0)}{s} \qquad \text{(VI-92)}$$

$$\lim_{t \to 0} i_o(t) = \lim_{s \to \infty} s\, i_o(s) = \lim_{s \to \infty} [\frac{\Delta V}{r + L_o s}] + i_o(0) = i_o(0), \qquad \text{(VI-93)}$$

and

$$\lim_{t \to \infty} i_o(t) = \lim_{s \to o} s\, i_o(s) = \frac{\Delta V}{r} + i_o(0) \qquad \text{(VI-94)}$$

but since

$$i_o(0) = \frac{V_o(0)}{r} \qquad \text{(VI-95)}$$

$$\lim_{\to \infty} i_o(t) = \frac{V_o(0) + \Delta V}{r} = \frac{V_o}{r} \qquad \text{(VI-96)}$$

which would be the new steady state value of i_o.

At this point, it would be well to examine (VI-71 through -76) and determine the initial value of the various flux linkages at $t \leq 0$, i.e., $\lambda_q(0)$, $\lambda_d(0)$, $\lambda_o(0)$, $\lambda_f(0)$, $\lambda_{kd}(0)$ and $\lambda_{kq}(0)$. To do this, consider that the machine, operating as a motor, is in a steady state situation at rated speed. Thus, all terms involving the time derivative, such as $d\lambda/dt$, are zero. Also, i_{kq}, i_{kd} are zero because there is no driving voltage for these currents other than induced transformer type voltages. Denoting t=0 values as (0), such as $i_f(0)$, $\lambda_q(0)$ etc. we have (with the above conditions),

$$\lambda_o(0) = L_o\, i_o(0) \qquad \text{(VI-97)}$$

$$V_o(0) = r\, i_o(0) \qquad \text{(VI-98)}$$

$$\therefore \lambda_o(0) = L_o\, i_o(0) \text{ or } \lambda_o = \frac{L_o V_o(0)}{r} \qquad \text{(VI-99)}$$

$$\lambda_d(0) = L_{df}\, i_f(0) + L_d i_d(0) \qquad \text{(VI-100)}$$

or:

$$V_q(0) = \omega \lambda_d(0) + r\, i_q(0) \qquad \text{(VI-101)}$$

from which:

$$\lambda_d(0) = V_q(0) - r\, i_q(0) \qquad \text{(VI-102)}$$

$$\lambda_q(0) = L_q\, i_q(0) \qquad \text{(VI-103)}$$

or:

$$V_d(0) = -\omega \lambda_d(0) + r\, i_d(0) \qquad \text{(VI-104)}$$

from which:

$$\lambda_q(0) = \frac{-V_d(0) + r\, i_d(0)}{\omega} \qquad \text{(VI-105)}$$

$$\lambda_{kq}(0) = L_{qg} i_q(0) \tag{VI-106}$$

$$\lambda_{kd}(0) = L_{df} i_f(0) + L_{df} i_d(0) \tag{VI-107}$$

$$V_f(0) = r_f i_f(0) \tag{VI-108}$$

$$\lambda_f(0) = L_f i_f(0) + L_{df} i_d(0) \tag{VI-109}$$

These equations give appropriate expressions for initial conditions.

Next, consider that we want λ_q as a function of i_q. From (VI-74,-75,-81) the appropriate equations are:

$$L_q i_q + L_{qg} i_{kq} = \lambda_q \tag{VI-110}$$

$$L_{qg} i_q + L_{kq} i_{kq} - \lambda_{kq} = 0 \tag{VI-111}$$

$$r_{kq} i_{kq} + \frac{d\lambda_{kq}}{dt} = 0 \tag{VI-112}$$

Taking the Laplace transform and rearranging to indicate $i_q(s)$ as the dependent variable yields: (Considering all variables are incremental values of the variable but not bothering to denote by Δ)

$$-\lambda_q(s) + L_{qg} i_{kq}(s) = -L_q i_q(s) \tag{VI-113}$$

$$L_{kq} i_{kq}(s) - \lambda_{kq}(s) = L_{qg} i_q(s) \tag{VI-114}$$

$$r_{kq} i_{kq}(s) + s\,\lambda_{kq}(s) = 0 \tag{VI-115}$$

from which:

$$\lambda_q(s) = \frac{\begin{vmatrix} -L_q i_q(s) & L_{qg} & 0 \\ -L_{qg} i_q(s) & L_{kq} & -1 \\ 0 & r_{kq} & +s \end{vmatrix}}{\begin{vmatrix} -1 & L_{qg} & 0 \\ 0 & L_{kq} & -1 \\ 0 & r_{kq} & +s \end{vmatrix}} \tag{VI-116}$$

$$\lambda_q(s) = \frac{L_q\left[1 + \frac{(L_q L_{kq} - L_{qg}^2)s}{r_{kq} L_q}\right] i_q(s)}{\left[1 + \frac{L_{kq}\, s}{r_{kq}}\right]} \qquad \text{(VI-117)}$$

now:

$$\frac{L_{kq} L_q - L_{qg}^2}{r_{kq} L_q} = \frac{L_{qg} + \ell_{kq} - \frac{L_{qg}^2}{L_q}}{r_{kq}} \qquad \text{(VI-118)}$$

$$= \frac{\ell_{kq} + \frac{L_{qg}(L_q - L_{gq})}{L_q}}{r_{kq}} \qquad \text{(VI-119)}$$

$$= \frac{\ell_{kq} + \frac{L_{qg}}{L_q} \ell_q}{r_{kq}} \qquad \text{(VI-120)}$$

Since:

$$L_{kq} = L_{qg} + \ell_{kq} \qquad \text{(VI-121)}$$

$$\ell_q = L_q - L_{qg} \qquad \text{(VI-122)}$$

We can define the following time constants:

$$T''_q = \frac{\ell_{kq} + \frac{L_{qg}\, \ell_q}{L_q}}{r_{kq}} \qquad \text{(VI-123)}$$

$$T''_{qo} = \frac{L_{kq}}{r_{kq}}$$

where

T''_{qo} = quadrature axis subtransient open circuit time constant

T''_q = quadrature axis subtransient short circuit time constant

T''_{qo} is the time constant one would see if the damper winding was opened and no coupling existed between the kq damper winding and any other winding. T''_q is the time constant associated with the damper winding if it is opened but coupied to the q winding, considered with zero resistance and short circuited. This can be seen if we write equations for our fictitious, opened kq winding with $r_q = 0$ and short circuited

$$V_{kq}(s) = (r_{kq} + L_{kq}\ s)i_{kq}(s) + L_{qg}s\ i_q(s) \qquad \text{(VI-124)}$$

$$0 = L_{qg}\ s\ i_{kq}(s) + L_q\ s\ i_q(s) \qquad \text{(VI-125)}$$

from which

$$i_{kq}(s) = \frac{V_{kq}(s)}{r_{kq}} \left[\frac{1}{1 + \frac{(L_q L_{kq} - L_{qg}^2)s}{L_q r_{kq}}} \right] \qquad \text{(VI-126)}$$

If ℓ_{kq} and ℓ_q from VI-121,-122 are used in (VI-123) for T''_q, we have

$$T''_q = \frac{L_q L_{kq} - L_{qg}^2}{r_{kq} L_q} \qquad \text{(VI-127)}$$

and it can be seen that the time constant associated with (VI-126) is the same as T''_q.

Hence the designation 'short circuit' as opposed to open circuit for T''_{qo}. The reason for calling it 'sub-transient' will be discussed in connection with the derivation of the d axis quantities. Now, returning to (VI-117) we can write this as:

$$\lambda_q(s) = L_q \frac{(1 + T''_q\ s)}{(1 + T''_{qo}\ s)} i_q(s) \qquad \text{(VI-128)}$$

The block diagram for this equation (recalling that $i_q(s)$ and $\lambda_q(s)$ are really $\Delta i_q(s)$ and $\Delta\lambda_q(s)$) is shown in Figure VI-8.

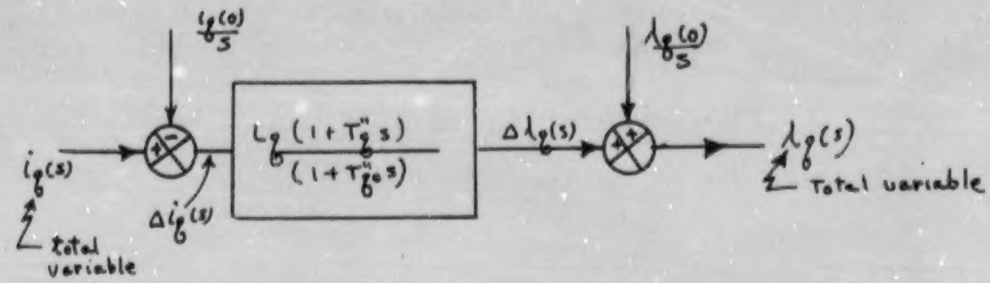

$\frac{i_q(0)}{s}$
$\frac{\lambda_q(0)}{s}$
$i_q(s)$
+
−
$\frac{L_q(1+T_q'' s)}{(1+T_{qo}'' s)}$
$\Delta \lambda_q(s)$
+
+
$\lambda_q(s)$
Total variable
$\Delta i_q(s)$
total variable

Figure VI-8. Quadrature Axis Block Diagram

We will now go through the same procedure for the d axis flux linkages, again using equations involving incremental changes until we get to the block diagram at which time we will denote initial values, total values and incremental values of the variables.

The equations relating events in the three windings on the direct axis are, from the equations for the synchronous machine,

$$L_d i_d + L_{df} i_f + L_{df} i_{kd} = \lambda_d \tag{VI-129}$$

$$L_{df} i_d + L_f i_f + L_{df} i_{kd} = \lambda_F \tag{VI-130}$$

$$L_{df} i_d + L_{df} i_f + L_{kd} i_{kd} = \lambda_{kd} \tag{VI-131}$$

$$V_f = \frac{d\lambda_f}{dt} + r_f i_f \tag{VI-132}$$

$$0 = \frac{d\lambda_{kd}}{dt} + r_{kd}\, i_{kd} \tag{VI-133}$$

From (VI-132) we can transform and solve for $_f(s)$. Thus

$$\lambda_f(s) = \frac{V_f(s) + r_f\, i_f(s)}{s} \tag{VI-134}$$

Transform (VI-130) and substitute into (VI-134) yielding:

$$s\, L_{df} i_d(s) + (r_f + s\, L_f) i_f(s) + L_{df}\, s\, i_{kd}(s) = V_f(s) \tag{VI-135}$$

Transform (VI-133), solve for $\lambda_{kd}(s)$, substitute in the transformed (VI-131) yielding:

$$s\ L_{df}\ i_f(s) + s\ L_{df}\ i_f(s) + (r_{kd} + s\ L_{kd})\ i_{kd}(s) = 0 \tag{VI-136}$$

(VI-129) is the remaining equation in the group. Transforming it, we have:

$$L_d\ i_d(s) + L_{df}\ i_f(s) + L_{df}\ i_{kd}(s) = \lambda_d(s) \tag{VI-137}$$

Although we want $\lambda_d(s)$ as the output of our block diagram with $i_d(s)$, $V_f(s)$ as inputs, it is less complicated to solve for $i_d(s)$ as a function of $\lambda_d(s)$, $V_f(s)$ and then rearrange to obtain $\lambda_d(s)$ as desired. Thus, from (VI-135, -136,-137) we have:

$$i_d(s) = \frac{\begin{vmatrix} V_f(s) & (r_f + L_f s) & L_{df}\ s \\ 0 & L_{df} s & (r_{kd} + L_{kd}\ s) \\ \lambda_d(s) & L_{df} & L_{df} \end{vmatrix}}{\begin{vmatrix} s\ L_{df} & (r_f + L_f s) & L_{df}\ s \\ s\ L_{df} & L_{df} s & (r_{kd} + L_{kd} s) \\ L_d & L_{df} & L_{df} \end{vmatrix}} = \frac{\Delta}{D} \tag{VI-138}$$

Solving first for Δ :

$$i_d(s)\ D = (r_f r_{kd})[1 + (\frac{L_f}{r_f} + \frac{L_{kd}}{r_{kd}})\ s + (\frac{L_f L_{kd}\ L_{df}^2}{r_f\ r_{kd}})\ s^2]\ \lambda_d(s) +$$

$$- (r_{kd} L_{df})[1 + \frac{L_{kd} - L_{df}}{r_{kd}}\ s]\ \ V_f(s) \tag{VI-139}$$

recall that:

$$L_f = L_{df} + \lambda_f \tag{VI-140}$$

$$L_{kd} = L_{df} + \lambda_{kd} \tag{VI-141}$$

and note that:

$$L_f L_{kd} = L_{df}^2 = (L_{df} + \ell_f)(L_{df} + \ell_{kd}) - L_{df}^2$$

$$= (\ell_f + \ell_{kd})L_{df} + \ell_f \ell_{kd}$$

$$= (L_{df} + \ell_f)\ell_{kd} + L_{df}\ell_f$$

$$= (L_{df} + \ell_f)(\ell_{kd} + \frac{L_{df}\ell_f}{L_{df} + \ell_f}) \qquad \text{(VI-142)}$$

from which:

$$\frac{L_f L_{kd} - L_{df}^2}{r_f\, r_{kd}} = (\frac{L_{df} + \ell_f}{r_f})(\frac{\ell_{kd} + \frac{L_{df}\ell_f}{L_{df} + \ell_f}}{r_{kd}}) \qquad \text{(VI-143)}$$

also:

$$\frac{L_{kd} - L_{df}}{r_{kd}} = \frac{L_{df} + \ell_{kd} - L_{df}}{r_{kd}} = \frac{\ell_{kd}}{r_{kd}} \qquad \text{(VI-144)}$$

and:

$$\frac{L_f - L_{df}}{r_f} = \frac{L_{df} + \ell_f - L_{df}}{r_f} = \frac{\ell_f}{r_f} \qquad \text{(VI-145)}$$

defining:

$$T_1 = \frac{L_f}{r_f} = \frac{L_{df} + \ell_f}{r_f} \qquad \text{(VI-146)}$$

$$T_2 = \frac{L_{kd}}{r_{kd}} \qquad \text{(VI-147)}$$

$$T_3 = \frac{\ell_{kd} + \frac{L_{df}\ell_f}{L_{df} + \ell_f}}{r_{kd}} \qquad \text{(VI-148)}$$

$$T_{kd} = \frac{\ell_{kd}}{r_{kd}} \qquad \text{(VI-149)}$$

$$T_{\ell F} = \frac{\ell_f}{r_f} \qquad \text{(VI-150)}$$

Using the defined quantities (VI-146 through VI-150) we can express (VI-139) as:

$$i_d(s)D = r_f e_{kd}[1 + (T_1+T_2)s + T_1T_3s^2]\lambda_d(s) - r_{kd}L_{df}[1+T_{kd}s]V_f(s) \qquad \text{(VI-151)}$$

If we examine the time constants defined above we note that: $T_1 >> T_2$ and $T_1 >> T_3$ because T_1 is associated with the field winding consisting of many turns in order to provide the necessary ampere turns of excitation and the main air gap flux whereas T_2 and T_3 are associated with the damper winding which is a cage of relatively few turns. We will introduce the approximation that;

$$T_1 + T_3 \approx T_1 + T_2 \qquad \text{(VI-152)}$$

from which:

$$1 + (T_1 + T_2)s + T_1T_3\, s^2 \approx (1 + T_1s)(1 + T_3s) \qquad \text{(VI-153)}$$

We will dignify T_1, T_3 by a formal name and a standard subscript. Thus:

$T_1 = T'_{do}$ = the direct axis transient open circuit time constant

$T_3 = T''_{do}$ = the direct axis subtransient open circuit time constant

T_{kd} = the direct axis damper leakage time constant

Note that T''_{do}, the "subtransient" open circuit time constant involves the damper winder, kd, and is thus much smaller than T'_{do} the "transient" open circuit time constant which is the field circuit considered above and is relatively long. Hence the name subtransient - to designate the first (and shortest lived) transient phenomena.

We can now write (VI-151) as:

$$i_d(s)D = r_f r_{kd}\,(1 + T'_{do}\,s)(1 + T''_{do}\,s)\;\lambda_d(s) - r_{kd}L_{df}(1 + T_{kd}s)\,V_f(s) \qquad \text{(VI-154)}$$

Now, the denominator, D, of (VI-138) is expanded, yielding:

$$D = L_d(r_f + s\,L_f)(r_{kd} + s\,L_{kd}) + s^2L^3_{df} + s^2L^3_{df} - s\,L^2_{df}\,(r_f + s\,L_f) +$$

$$- s^2L^2_{df}\,L_d - s\,L^2_{df}\,(r_{kd} + s\,L_{kd}) \qquad \text{(VI-155)}$$

$$D = L_d[r_f r_{kd}] + L_d[r_{kd}(L_{df} + \ell_f) + r_f(L_{df} + \ell_{kd}) - r_f \frac{L_{df}^2}{L_d} - r_{kd} \frac{L_{df}^2}{L_d}]\, s +$$

$$+ [L_d L_f L_{kd} + 2 L_{df}^3 - L_{df}^2 L_f - L_{df}^2 L_d - L_{df}^2 L_{kd}] s^2 \qquad \text{(VI-156)}$$

$$= L_d r_f r_{kd} \left[1 + \left\{ \frac{(L_{df} + \ell_f - \frac{L_{df}^2}{L_d})}{r_f} + \frac{(L_{df} + \ell_{kd} - \frac{L_{df}^2}{L_d})}{r_{kd}} \right\} s + \right.$$

$$\left. + \frac{1}{r_f r_{kd}} \left(\frac{L_d L_f L_{kd} + 2 L_{df}^3 - L_{df}^2 L_f - L_{df}^2 L_d - L_{df}^2 L_{kd}}{L_d} \right) s^2 \right] \qquad \text{(VI-157)}$$

Now,

$$L_{df} + \ell_f - \frac{L_{df}^2}{L_d} = \frac{(L_{df} + \ell_f)(L_{df} + \ell_d) - L_{df}^2}{L_d}$$

$$= \frac{(L_{df} + \ell_d)\ell_f + \ell_d L_{df}}{L_{df} + \ell_d} = \ell_f + \frac{\ell_d L_{df}}{L_{df} + \ell_d} \qquad \text{(VI-158)}$$

Similarly

$$L_{df} + \ell_{kd} - \frac{L_{df}^2}{L_d} = \ell_{kd} + \frac{\ell_d L_{df}}{L_{df} + \ell_d} \qquad \text{(VI-159)}$$

The coefficient of s^2 can be simplified as follows:

$$\frac{1}{r_f r_{kd}} [L_d L_f L_{kd} + 2L_{df}^3 - L_{df}^2 (L_f + L_d + L_{kd})] =$$

$$= \frac{1}{r_f r_{kd}} [(L_{df} + \ell_d)(L_{df} + \ell_f)(L_{df} + \ell_{kd}) + 2L_{df}^3 - L_{df}^2(L_{df} + \ell_f) - L_{df}^2(L_{df} + \ell_d) +$$

$$- L_{df}^2 (L_{df} + \ell_{kd})]$$

$$= \frac{1}{r_f r_{kd}} [L_{df}(\ell_f \ell_{kd} + \ell_d \ell_f + \ell_d \ell_{kd}) + \ell_d \ell_f \ell_{kd}] \qquad \text{(VI-160)}$$

In order to determine a product which yields this - and to make one of the terms of this product the same as one term of the sum which forms the coefficient of s, divide the bracketed portion of (VI-160) by (VI-159) noting that:

$$\ell_f + \frac{\ell_d L_{df}}{L_{df} + \ell_d} = \frac{\ell_d \ell_f + (\ell_d + \ell_f) L_{df}}{L_d} \qquad \text{(VI-161)}$$

Thus:

$$\begin{array}{r l} & L_d \ell_{kd} \\ \cline{2-2} \dfrac{\ell_d \ell_f + (\ell_d + \ell_f) L_{df}}{L_d} & \Big)\, (\ell_d \ell_f \ell_{kd} + L_{df}(\ell_f \ell_{kd} + \ell_d \ell_f + \ell_d \ell_{kd}) \end{array}$$

$$\frac{\ell_d \ell_f \ell_{kd} + L_{df}(\ell_f \ell_{kd} + \ell_d \ell_{kd})}{L_{df}\, \ell_d \ell_f} \qquad \text{(VI-162)}$$

we can express the result of the division operation in (VI-162) as:

$$L_d \ell_{kd} + \frac{L_{df} \ell_d \ell_f L_d}{\ell_d \ell_f + (\ell_d + \ell_f) L_{df}} \qquad \text{(VI-163)}$$

The coefficient of s^2 in (VI-155) is then:

$$\frac{L_d}{r_f r_{kd}} \left[\left(\ell_{kd} + \frac{L_{df} \ell_d \ell_f}{(L_{df} \ell_d + \ell_f \ell_d + L_{df} \ell_f)}\right)\left(\ell_f + \frac{\ell_d L_{df}}{L_{df} + \ell_d}\right)\right] \qquad \text{(VI-164)}$$

Using (VI-158,-159,-164) we can define the following time constants:

$$T_4 = \frac{\ell_f + \dfrac{\ell_d L_{df}}{L_{df} + \ell_d}}{r_f} \qquad \text{(VI-165)}$$

$$T_5 = \frac{\ell_{kd} + \dfrac{\ell_d L_{df}}{L_{df} + \ell_d}}{r_{kd}} \qquad \text{(VI-166)}$$

$$T_6 = \ell_{kd} + \frac{L_{df}\ell_d\ell_f}{L_{df}\ell_d + \ell_f\ell_d + L_{df}\ell_f} \qquad \text{(VI-167)}$$

With these time constants, (VI-157) becomes

$$D = L_d r_f r_{kd} \left[1 + (T_4 + T_5)s + T_4\, T_6 s^2\right] \qquad \text{(VI-168)}$$

Again using the argument that time constants associated with the field are much larger than those associated with the damper circuits we can assume:

$$T_4 \gg T_5 \text{ and } (T_4 + T_6) \approx (T_4 + T_5) \qquad \text{(VI-169)}$$

we can express (VI-168) as the approximation:

$$D \approx L_d r_f r_{kd}\,(1 + T_4 s)(1 + T_6 s) \qquad \text{(VI-170)}$$

We will denote the time constants T_4 and T_6 as:

$T_4 = T'_d$ = direct axis short circuit transient time constant

$T_6 = T''_d$ = direct axis short circuit subtransient time constant

and we can then express (VI-168) as:

$$D \approx L_d r_f r_{kd}\,\left[(1 + T'_d s)(1 + T''_d s)\right] \qquad \text{(VI-171)}$$

From (VI-165,-166,-167,-154) we have:

$$L_d r_f r_{kd}(1 + T'_d s)(1 + T''_d s) i_d(s) = r_f r_{kd}(1 + T'_{do} s)(1 + T''_{do} s)\lambda_d(s) +$$
$$-r_{kd} L_{df}\,(1 + T_{kd} s)\, V_f(s) \qquad \text{(VI-172)}$$

Solving (VI-172) for $\lambda_d(s)$ yields:

$$\lambda_d(s) = \frac{L_d(1 + T'_d s)(1 + T''_d s)}{(1 + T'_{do} s)(1 + T''_{do} s)}\, i_d(s) + \frac{L_{df}\,(1 + T_{kd} s)}{r_f(1 + T'_{do} s)(1 + T''_{do} s)}\, V_f(s) \qquad \text{(VI-173)}$$

($\lambda_d(s)$, $i_d(s)$ and $V_f(s)$) in (VI-173) are incremental values as discussed previously. In block diagram form:

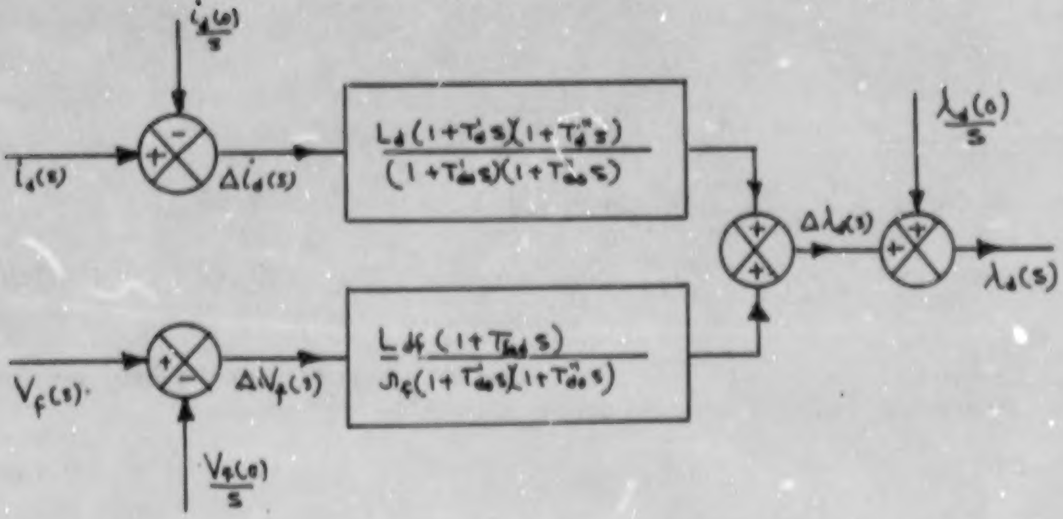

$\frac{i_d(0)}{s}$
$i_d(s)$
$\Delta i_d(s)$
$\frac{L_d(1+T'_d s)(1+T''_d s)}{(1+T'_{do} s)(1+T''_{do} s)}$
$\frac{\lambda_d(0)}{s}$
$\Delta\lambda_d(s)$
$\lambda_d(s)$
$V_f(s)$
$\Delta V_f(s)$
$\frac{L_{df}(1+T_{kd} s)}{r_f(1+T'_{do} s)(1+T''_{do} s)}$
$\frac{V_f(0)}{s}$
+
-

Figure VI-9. Direct Axis Block Diagram

We are now in a position to summarize the equations we have developed and to present additional defined reactances. The fundamental quantities are:

r = armature resistance

r_f = field resistance

r_{kd} = direct axis damper effective resistance

r_{kq} = quadrature axis damper effective resistance

L_{df} = direct axis mutual inductance

L_{qg} = quadrature axis mutual inductance

$\ell_d = \ell_q = \ell_a$ = armature leakage (d winding, q winding) inductance

ℓ_f = field leakage (F winding) inductance

ℓ_{kq} = quadrature axis damper (KQ) winding leakage inductance

ℓ_{kd} = direct axis damper (KD) winding leakage inductance

$L_q = L_{qg} + \ell_a$ = quadrature axis synchronous reactance

$L_d = L_{df} + \ell_a$ = direct axis synchronous reactance

We can also define additional inductances as follows:

$$L'_d = \ell_a + \frac{L_{df}\ell_f}{L_{df} + \ell_f} = \text{direct axis transient reactance}$$

$$L''_d = \ell_a + \frac{L_{df}\ell_f\ell_{kd}}{L_{df}\ell_f + \ell_f\ell_{kd} + L_{df}\ell_{kd}} = \text{direct axis subtransient reactance}$$

$$L''_q = \ell_a + \frac{L_{qg}\ell_{kq}}{L_{qg} + \ell_{kq}} = \text{quadrature axis subtransient reactance}$$

The following time constants were defined:

$$T'_{do} = \frac{L_{df} + \ell_f}{r_f} = \text{direct axis transient open circuit time constant}$$

$$T'_d = \frac{\ell_f + \frac{L_{df}\ell_a}{L_{df} + \ell_a}}{r_f} = \text{direct axis transient short circuit time constant}$$

$$T''_{do} = \frac{1}{r_{kd}}(\ell_{kd} + \frac{L_{df}\ell_f}{L_{df} + \ell_f}) = \text{direct axis subtransient open circuit time constant}$$

$$T''_d = \frac{1}{r_{kd}}(\ell_{kd} + \frac{L_{df}\ell_a\ell_f}{L_{df}\ell_a + L_{df}\ell_f + \ell_f\ell_a}) = \text{direct axis subtransient short circuit time constant}$$

$$T''_{qo} = \frac{1}{r_{kq}}(L_{qg} + \ell_{kq}) = \text{quadrature axis subtransient open circuit time constant}$$

$$T''_q = \frac{1}{r_{kq}}(\ell_{kq} + \frac{L_{qg}\ell_a}{L_{qg} + \ell_a}) = \text{quadrature axis subtransient short circuit time constant}$$

$$T_{kd} = \frac{kd}{r_{kd}} = \text{direct axis damper leakage time constant}$$

Other relationships relating L_d, L_q and various time constants to L'_d, L''_d, L''_q can be obtained by noting that:

$$L_d \frac{T'_d}{T'_{do}} = (\ell_a + L_{df}) \; \frac{\frac{1}{r_f}\left(\ell_f + \frac{L_{df}\ell_a}{L_{df} + \ell_a}\right)}{\frac{1}{r_f}(\ell_f + L_{df})} = \frac{(L_{df} + \ell_a)\ell_f + L_{df}\ell_a}{L_{df} + \ell_f} \qquad \text{(VI-174)}$$

$$= \frac{\ell_a\,(L_{df} + \ell_f) + L_{df}\ell_r}{L_{df} + \ell_f} = \ell_a + \frac{L_{df}\ell_f}{L_{df} + \ell_f} \qquad \text{(VI-175)}$$

$$= L'_d \qquad \text{(VI-176)}$$

$$\therefore\; L'_d = L_d \frac{T'_d}{T'_{do}} \qquad \text{(VI-177)}$$

Similarly

$$L''_d = L_d \frac{T'_d\; T''_d}{T'_{do}\; T''_{do}} \qquad \text{(VI-178}$$

$$L''_q = L_q \frac{T''_q}{T''_{qo}} \qquad \text{(VI-179)}$$

These last relationships are helpful when the machine constants are determined by test. They are also useful in analog simulation and in estimating short circuit levels for synchronous generator.

The equations, as developed are now:

$$v_d = \frac{d\lambda_d}{dt} - \omega\lambda_q + r\, i_d \qquad \text{(VI-180)}$$

$$v_q = \frac{d\lambda_q}{dt} + \omega\lambda_d + r\, i_q \qquad \text{(VI-181)}$$

$$v_o = \frac{d\lambda_o}{dt} + r\, i_o \qquad \text{(VI-182)}$$

$$\lambda_o = L_o i_o \qquad \text{(VI-183)}$$

$$\lambda_q(s) = X_1(s)\;\; i_q(s) \qquad \text{(VI-184)}$$

$$\lambda_d(s) = X_2(s)\; i_d(s) + X_3(s)\;\; e_f(s) \qquad \text{(VI-185)}$$

where

$$X_1(s) = L_q \frac{(1 + T''_q s)}{(1 + T''_{qo} s)} \qquad \text{(VI-186)}$$

$$X_2(s) = L_d \frac{(1 + T'_d s)(1 + T''_d s)}{(1 + T'_{do} s)(1 + T''_{do} s)} \qquad \text{(VI-187)}$$

$$X_3(s) = \frac{L_{df}}{r_f} \; \frac{(1 + T_{kd}\, s)}{(1 + T'_{do} s)(1 + T''_{do}\, s)} \qquad \text{(VI-188)}$$

and

$$T_e = T_L(\omega) + J \frac{d\omega}{dt} \qquad \text{(VI-189)}$$

$$T_e = k(i_q \lambda_d - i_d \lambda_q) \qquad \text{(VI-190)}$$

In order to evaluate k, recall that the instantaneous power converted to mechanical form is given by

$$p = v_a i_a + v_b i_b + v_c i_c \qquad \text{(VI-191)}$$

We can eliminate the phase quantities by the use of the transformation given in (VI-34). When this is done, the instantaneous power becomes

$$p = \frac{3}{2}(v_d i_d + v_q i_q + 2\, v_o i_o) \qquad \text{(VI-192)}$$

or, using the relationships for d,q, voltages

$$p = \frac{3}{2}[i_d \frac{d\lambda_d}{dt} - \omega\lambda_q i_d + r\, i_d^2 + i_q \frac{d\lambda_q}{dt} + \omega\lambda_d i_q + r\, i_q^2 + 2\, i_o \frac{d\lambda_o}{dt} + 2\, i_o^2 r] \qquad \text{(VI-193)}$$

Recall that terms such as $i \frac{d\lambda}{dt}$ represent a change in stored energy and that the $i^2 r$ terms are ohmic loss. The remaining terms represent power conversion from electrical to mechanical form. Therefore, shaft power, p_s, is given by:

$$p_s = \frac{3}{2}\omega(\lambda_d i_q - \lambda_q i_d) \qquad \text{(VI-194)}$$

and

$$T_e = \frac{p_s}{\omega} = \frac{3}{2}(\lambda_d i_q - \lambda_q i_d) \qquad \text{(VI-195)}$$

$$\therefore\ k = \frac{3}{2} \qquad \text{(VI-196)}$$

We now have the describing equations for the synchronous motor. The equations for the generator, or alternator, are obtained by changing the sign of the current. The equations are based on a transformation of variables from phase quantities to d-q-o quantities. This change in variable has enabled us to replace time varying inductances by constant value inductances (neglecting saturation effects) and our ability to simulate the machine by computer for dynamic studies is enhanced.

To use the equations, one must study the problem to be solved,adapt the equations to the specific problem, introduce the initial conditions and restraints and proceed either by computer or by the application of operationsal mathematical techniques. In the subsequent sections we will examine the behavior of the synchronous machine under various conditions in order to illustrate problem solving procedures.

VI.7 THE SYNCHRONOUS MACHINE UNDER STEADY STATE CONDITIONS: By means of the transformation used to convert phase quantities to d-q-o quantities we were able to define fictitious inductances L_o,L_d, and L_q (VI-39, -40, and -41) to replace the actual inductances in the machine, which are functions of angular position of the rotor and thus (since the rotor revolves) also functions of time. We will now apply the equations we have derived to the machine operating in the steady state:

For the general case;

$$\theta = \omega t + \alpha \tag{VI-197}$$

where:

θ = angle between the rotor axis (direct axis) and the axis of phase "a"

α = rotor position at $t = 0$.

Since α is completely arbitrary, we will assume $\alpha = 0$ for this analysis. This will not result in any loss of generality. With this assumption

$$\theta = \omega t \tag{VI-198}$$

The phase currents in the stator windings of a 3 phase machine can be expressed as

$$i_a = \sqrt{2}\ I \cos(\omega t + \beta) \tag{VI-199}$$

$$i_b = \sqrt{2}\ I \cos(\omega t + \beta - 120) \tag{VI-200}$$

$$i_c = \sqrt{2}\ I \cos(\omega t + \beta - 240) \tag{VI-201}$$

where:

I = root mean square value of the balanced phase currents
β = the phase angle of i_a with respect to the time origin,
i.e., at $t = 0$; $i_a = \sqrt{2}\, I \cos \beta$.

The d-q-o currents, corresponding to the actual phase currents are determined from (VI-199,-200,201) by using the transformation (VI-34). Thus:

$$i_d = \sqrt{2}\, I \cos \beta \qquad \text{(VI-202)}$$

$$i_q = \sqrt{2}\, I \sin \beta \qquad \text{(VI-203)}$$

$$i_o = 0 \qquad \text{(VI-204)}$$

The fact that these currents are not time varying is completely proper because these currents are associated with D, Q windings which (since the inductance relationships associataed with these windings is constant) have constant position with respect to the field, (F) winding. Since the i_d, i_q, i_o currents result in the same mmf as the phase currents which yield a constant mangitude mmf revolving at the same speed as the rotor it is necessary that the d,q,o currents have constant magnitude to yield, also, a constant mmf. This is true only for steady state conditions.

Physical insight into what has been accomplished can be gained by considering that in the actual machine we have stationary stator windings which have a net mmf which revolves with the same speed as the rotor. If an observer is on the revolving rotor and examined the mmf of the stator, he could very easily assume that this mmf resulted from a coil rotating at rotor speed and at some fixed angular displacement from the axis of the rotor. Or, he could assume that the mmf resulted from two coils, revolving at rotor speed, (one along the axis of the rotor and one in quadrature with the rotor) whose net mmf was such that it yielded the proper magnitude and phase relationships of stator mmf with respect to the rotor.

From (VI-36,-37, and -38) it can be seen that if the various currents are either zero or of constant value the air gap flux linkages are also constant.

Since we are assuming steady state conditions in this particular analysis

$$i_{kd} = 0; \; i_{kq} = 0 \qquad \text{(VI-205)}$$

because the flux linkages linking the KD, KQ windings are constant.

$$i_f = \text{constant} \qquad \text{(VI-206)}$$

because of constant excitation (steady state)

$$i_o = 0 \qquad \text{(VI-207)}$$

i_d, i_q are constant from [VI-202,-203, and -204].

If flux linkages are constant, the terms involving transformer voltages (based on time rate of change of flux linkages) are zero.

Using the above, we can write the equations for a generator, or alternator, in the steady state as: (From (VI-180) through (VI-190) with currents i_d and i_q having opposite sign because these equations describe motor action).

$$v_d = -\omega\lambda_q - r\, i_d \qquad \text{(VI-208)}$$

$$v_q = \omega\lambda_d - r\, i_q \qquad \text{(VI-209)}$$

$$v_f = r_f i_f \qquad \text{(VI-210)}$$

$$\lambda_q = -L_q i_q \qquad \text{(VI-211)}$$

$$\lambda_d = L_{df} i_f - L_d i_d \qquad \text{(VI-212)}$$

$$\lambda_f = L_f i_f - L_{df}\, i_d \qquad \text{(VI-213)}$$

$$T_e = \frac{3}{2}\left(i_q \lambda_d - i_d \lambda_q\right) \qquad \text{(VI-214)}$$

Construction of a phasor diagram is helpful in visualizing the steady state operating condition. Note that in (VI-198), $\theta = \omega t$. Using this in the inverse transformation, (VI-35), for voltage, rather than current, the "a" phase voltage is written as:

$$v_a(t) = v_d \cos \omega t - v_q \sin \omega t \qquad \text{(VI-215)}$$

Now, $-\sin \omega t = \cos(\omega t + 90)$. Therefore:

$$v_a(t) = v_d \cos \omega t + v_q \cos(\omega t + 90) \qquad \text{(VI-216)}$$

When we construct a phasor diagram, our choice of time is completely arbitrary. Therefore, we will choose $t = 0$ for simplicity, yielding from (VI-216):

$$v_a = v_d + j\, v_q \qquad \text{(VI-217)}$$

From (VI-216), $v_a(t)$ is the instantaneous terminal voltage of "a" phase and therefore v_d and v_q correspond to maximum values. Therefore v_a is the maximum value of "a" phase voltages, with our time reference. If we denote the rms terminal voltage as V_{ta}, we have:

$$V_{ta} = \frac{v_a}{\sqrt{2}} = \frac{v_d}{\sqrt{2}} + j\,\frac{v_q}{\sqrt{2}} = V_d + j\, V_q \qquad \text{(VI-218)}$$

where V_d and V_q correspond to rms values. Similarly:

$$I_a = \frac{i_d}{\sqrt{2}} + j\,\frac{i_q}{\sqrt{2}} = I_d + j\,I_q \tag{VI-219}$$

Now, referring to (VI-208,-209) with the values of flux linkages from (VI-211,-212) substituted, we have:

$$v_d = \omega L_q i_q - r\,i_d \tag{VI-220}$$

$$v_q = \omega L_{df} i_f - \omega L_d i_d - r\,i_q \tag{VI-221}$$

Let:

$$\omega L_q = X_q \tag{VI-222}$$

$$\omega L_d = X_d \tag{VI-223}$$

substitute (VI-220) and (VI-221) into (VI-218) and divide every item by $\sqrt{2}$, yielding:

$$\frac{v_a}{\sqrt{2}} = \left(X_q\,\frac{i_q}{\sqrt{2}} - r\,\frac{i_d}{\sqrt{2}}\right) + j\left(\frac{\omega L_{df} i_f}{\sqrt{2}} - X_d\,\frac{i_d}{\sqrt{2}} - \frac{r\,i_q}{\sqrt{2}}\right) \tag{VI-224}$$

$$V_{ta} = \frac{v_a}{\sqrt{2}} = X_q I_q - j\,X_d I_d + j\,E_f - (I_d + j\,I_q)r \tag{VI-225}$$

Note that V_{ta}, I_q, I_d and $I_a = I_d + j\,I_q$ are rms values and;

$$E_f = \frac{\omega L_{df}\,i_f}{\sqrt{2}} \tag{VI-226}$$

E_f is the rms voltage that would appear at the machine terminals if i_d and I_q were zero - in other words the "open circuit" voltage and it is due to field excitation only. It is commonly referred to as the "excitation voltage".

(VI-225) can be rearranged as (in rms values)

$$V_{ta} = (I_q X_q - I_a r) + j\,(E_f - I_d X_d) \tag{VI-227}$$

Recognizing that V_{ta} and I_a are phasors, we can rewrite (VI-227) as

$$V_{ta} + I_a r + j\, I_d X_d - X_q I_q = j\, E_f \qquad \text{(VI-228)}$$

A phasor diagram can be constructed from (VI-228). To draw the phasor diagrams, we will first draw in the direct and quadrature axis (the q axis 90° ahead of the d axis) and consider the d axis as the 'real' axis and the q axis as the imaginary axis. Thus, if we plot 'real' quantities, from above, along the d axis, we will plot the imaginary quantities (rotated 90° ahead because of the j operator) along the q axis. For convenience (and to conform to rather widespread custom) we will locate the q axis horizontally. Doing this, we have the phasor diagram shown in Figure VI-10.

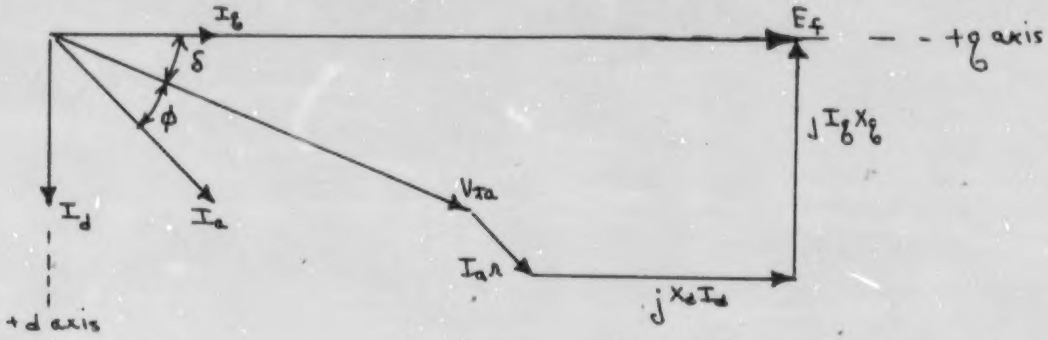


Figure VI-10. Phasor Diagram, Generator

Note that:

a) $I_q X_q$ is in the negative 'real' direction,

b) $I_q X_q$ and $I_d X_d$ are voltage drops across an inductance and they lead the currents I_q and I_d by 90° respectively. For this reason it is customary to denote these voltage drops by the operator j to indicate the phase relationship with current.

Now, suppose we are given a specific situation at the terminals, i.e., an armature phase current, I_a, a terminal voltage, V_t, and a power factor angle, ϕ. We do not know δ and thus cannot resolve I_a into components I_d and I_q which are necessary to proceed with the construction of the phasor diagram. This difficulty can be curcumvented by a relationship we can obtain from Figure VI-11.

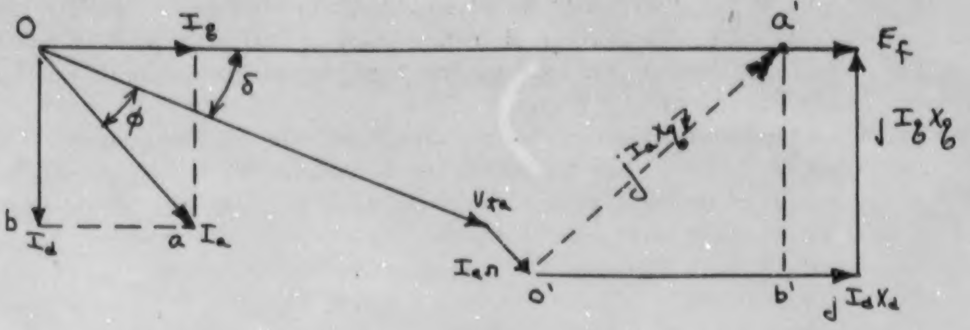


Figure VI-11. To locate point a'

From geometrical considerations:

$$\frac{0'a'}{0a} = \frac{b'a'}{ba} \tag{VI-229}$$

but:

$$0a = I_a \text{ ; } b'a' = j\ I_q X_q ; \ ba = I_q$$

$$\therefore o'a' = oa\ \frac{b'a'}{ba} = (I_a)\left(\frac{j\ I_q X_q}{(I_q)}\right) = j\ I_a\ X_q \tag{VI-230}$$

Therefore, we can proceed as follows: V_{ta}, and I_a separated by ϕ. Draw $I_a r$ parallel (in phase) with I_a. Draw o'a' perpendicular to the direction of I_a. This locates point a'. Draw a line through points 0–a'. Resolve I_a into I_q, I_d and construct $j\ I_d X_d$, $j\ I_q X_q$. This locates E_f which in turn determines the necessary excitation for a given machine - load situation. A relationship for δ in a motor is the subject of a homework problem.

In the situation considered above it should be noted that we can control the magnitude of E_f by changing field excitation, i_f. Suppose our alternator is paralleled with another machine of size, or rating, much larger than the machine under consideration. (or in other, more common terms, our machine is connected to an "infinite bus" which may well mean that we are looking at one machine in a very large system). If the capacity paralleled with the machine under consideration is very large, the terminal voltage, V_{ta}, will be determined by the large capacity system and will remain constant regardless of the magnitude of E_f of the machine under study. The current supplied by our machine will change its position with respect to V_{ta} in order to satisfy the mathematical relationship (in effect, Kirchoff's law). Under some conditions of magnitude of E_f, the current, I_a, may lead the voltage, V_{ta}. Thus, with an alternator paralleled to an infinite bus, we can cause that machine to supply current at either lagging, unity or leading power factor into the infinite bus (even though the load on the total system may require net current at opposite power factor conditions). Of course, if the net current to the system load was lagging power factor and some machine (or machines) delivered leading current, other machines would have to "absorb" this by delivering current at a lagging power factor in excess of net system power factor. Figure VI-12 portrays an alternator with reduced excitation. Note that, the current now leads the voltage. This condition, for an alternator, is referred to as "under-excited" whereas the lagging power factor condition is referred to as "over-excited". The distinction between "under" and "over" excited occurs at the excitation corresponding to unity power factor.

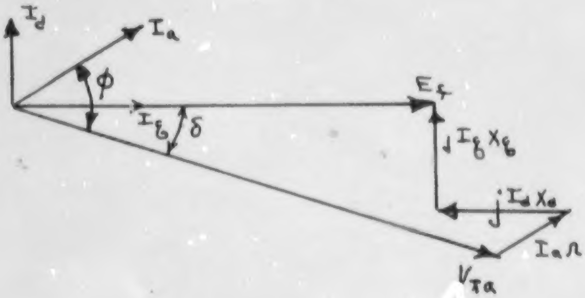


Figure VI-12. Phasor Diagram, Under-Excited Alternator

Note that, in each diagram, V_{ta} lags E_f by an angle δ. δ is the "power angle" or torque angle" previously encountered in connection with the reluctance motor. We will examine it in more detail in the next section. For now, note that E_f will lag V_{ta} in the phasor diagram for a synchronous motor. To develop the phasor diagram for the motor we will reverse the direction of I_a with respect to V_{ta}, (corresponding to a reversal of power flow) used in the alternator phasor diagram . The diagram for an over excited synchronous motor is shown in Figure VI-13.

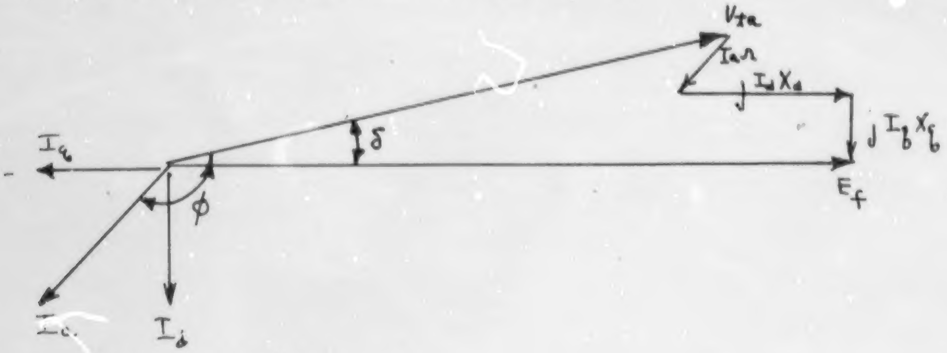


Figure VI-13. Phasor Diagram, Over-Excited Motor

The phasor diagram for an under-excited synchronous motor is shown in Figure VI-14.

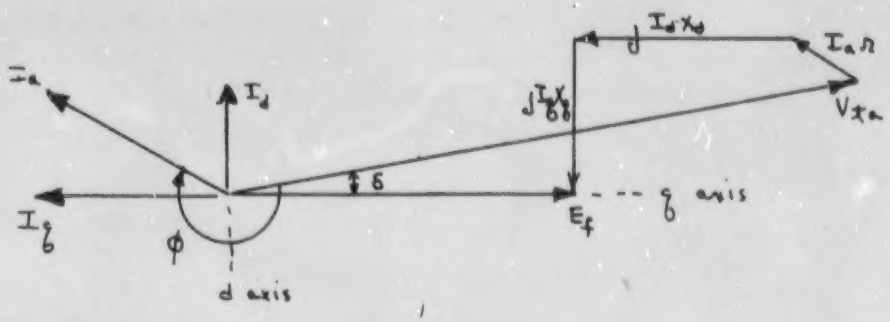


Figure VI-14. Phasor Diagram, Under-Excited Motor

In order to determine, quantiatively, the power factor relationships for the motor, compare Figure VI-12 and VI-13. The phase relationship between V_{ta} and I_a is perserved (except for direction of power flow). However, E_f for the motor is much larger than for the same power factor relationship for the alternator. In each case, then, the power factor is leading. However the alternator is under excited; the motor is over excited. By the same token, an over excited alternator, delivers lagging power factor current into an infinite bus system whereas an under excited motor absorbs power at lagging power factor.

This ability to vary the phase relationship between current and voltage by manipulating the excitation is a major advantage enjoyed by a synchronous motor over an asynchronous, or induction, motor. Many industrial firms purchase energy from a supplier (an electric utility) under a cost contract which provides for a more favorable (to the customer) rate for energy used at near unity power factor. In practice, the majority of loads are inherently inductive. The power factor of energy taken by the industrial customer can be improved (increased) by either installation of static capacitors or by providing synchronous motors operating over excited and taking energy at leading power factor.

A synchronous motor develops average torque only at synchronous speed (when the rotor field and stator field are stationary with respect to each other). Thus, it has no starting torque as a synchronous motor, per se. However, the damper windings, KQ and KD, (or the F winding if short circuited) do behave in the same fashion as the rotor winding of a squirrel cage induction motor and do provide an asynchronous torque for starting. When the motor nears synchronous speed (it can never reach synchronous speed running as an asynchronous motor because as the mechanical speed approaches synchronous the relative motion between the rotor windings and the stator field approach zero, no flux is cut, no voltage and consequent currents in KD and KQ are available and the torque goes to zero) the field winding is energized and the synchronous motor "pulls into step", i.e., runs synchronously.

The necessity for control circuitry to apply the field excitation at precisely the proper time and the requirement to provide a source of d.c. for excitation (usually from a small d.c. generator which is direct shaft connected to the motor and called an "exciter") cause the synchronous motor to be more expensive than an induction motor of the same power rating. An exception to this is for motors to run at relatively slow speeds. For the general case, however, the economics may very well be in favor of the synchronous motor when its power factor correcting capability is evaluated economically against an induction motor plus static capacitors.

Synchronous motors are designated as having "unity power factor" or "0.8 power factor" capability. The 0.8 p.f. refers to leading power factor capability. These ratings mean that the field winding resistance is low enough so that excitation current necessary for that power factor at rated shaft power output can be obtained without exceeding the thermal rating of the machine. Since 0.8 power factor leading rating involves higher excitation level than does the unity power factor rating ("over excitation") a larger exciter and lower resistance (more copper) field winding are required. Thus, it costs more. Each application is evaluated on its own economic merits.

VI.8 THE POWER ANGLE AND POWER CONVERTED, STEADY STATE: In Section VI.7 we encountered the power, or torque, angle, δ, which is the angle between the terminal voltage of "a" phase and the axis of the field mmf (also denoted as E_f). This angle has a definite relationship to the magnitude of torque and power resulting from the conversion process. To analytically investigate this, we will consider a synchronous alternator paralleled, through an impedance, with an infinite bus. The external impedance may very well represent a transformer changing the alternator voltage level to that of the transmission system to which the alternator is supplying energy. For large machines, the ratio of reactance to resistance of machines and transformers is usually greater than 10. Therefore, it seems valid to neglect resistance in our derivation. The phasor diagram for such a configuration involving an over excited alternator is shown in Figure VI-15.

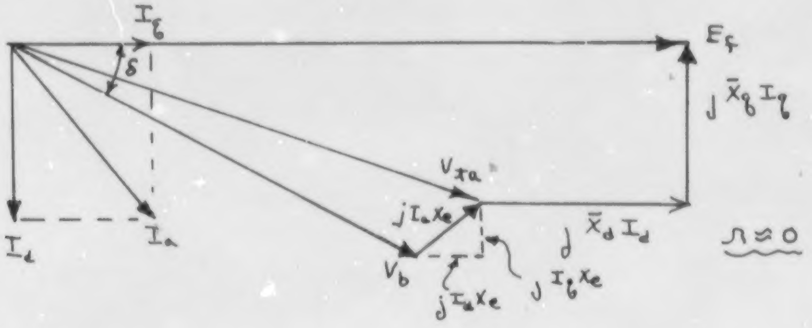


Figure VI-15. Phasor Diagram, Alternator Connected to Infinite Bus through Series Reactance $r \approx 0$

The machine direct and quadrature axis reactance is designated as $\overline{X}_d$ and $\overline{X}_q$ in order to distinguish them from total reactances as calculated below. Note that the infinite bus voltage is denoted as V_b and that it differs from V_{ta}, the alternator terminal voltage by the voltage drop $j\ I_a X_e$ which can be resolved into components $j\ I_d X_e$ and $j\ I_q X_e$. We are considering the power transfer from the shaft input to the power input to the infinite bus. Thus we are concerned with the angle δ existing between V_b and E_f and we can consider the reactance X_e to be a part of the total reactance of a machine connected to the infinite bus by adding X_e to the reactances $\overline{X}_d$ and $\overline{X}_q$ of the machine. The phasor diagram then becomes:

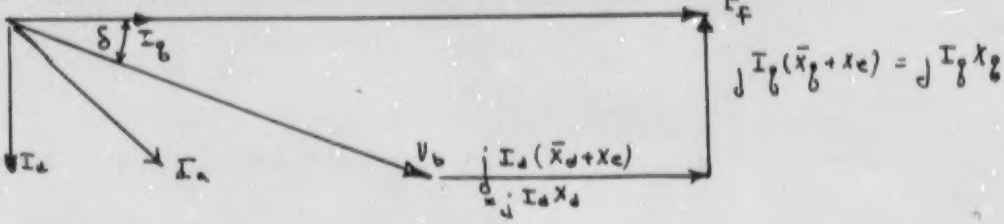

$\bar{E}_f$
δ
I_q
$j\,I_q(\bar{X}_q + X_e) = j\,I_q X_q$
I_d
I_a
V_b
$j\,I_d(\bar{X}_d + X_e)$
$= j\,I_d X_d$

Figure VI-16. Phasor Diagram of Figure VI-15 with X_e Included with Machine Reactances

Recall that power is the product of the in-phase components of current and voltage. Thus:

$$\frac{P}{3} = I_d V_b \sin\delta + I_q V_b \cos\delta \qquad \text{(VI-231)}$$

I_d and I_q can be determined from Figure VI-16 as:

$$I_d = \frac{E_f - V_b \cos\delta}{X_d} \qquad \text{(VI-232)}$$

and:

$$I_q = V_b \frac{\sin\delta}{X_q} \qquad \text{(VI-233)}$$

Substituting (VI-232,-233) in (VI-231) yields:

$$\frac{P}{3} = \frac{E_f V_b}{X_d} \sin\delta + \frac{V_b^2}{2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right) \sin 2\delta \qquad \text{(VI-234)}$$

This relationship is shown in Figure VI-17.

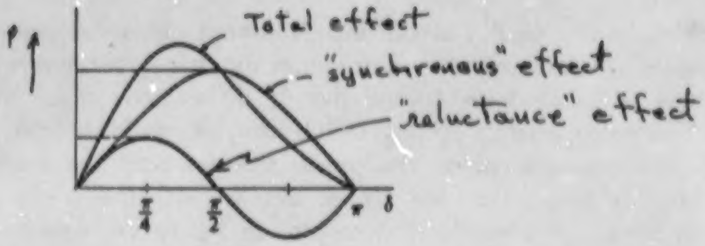


Figure VI-17. Synchronous Machine Power Angle Characteristic

Some synchronous machines have a rotor structure which is cylindrical (notably two and four pole high speed alternators for steam turbine drive service) and the rotor in these machines presents uniform (regardless of rotor position) magnetic reluctance to flux. For this situation, $X_d = X_q = X_s$ and the power expression, corresponding to (VI-194), is:

$$\frac{P}{3} = \frac{E_f V_b}{X_s} \sin \delta \qquad \text{(VI-235)}$$

This is, in fact, the relationship existing for power transfer between two voltages connected through a reactance. If the reactance were replaced by an impedance consisting of an inductive reactance, X, and resistance R it can be shown that the power transfer relationships are:

$$\frac{P_s}{3} = \frac{E_f V_b}{Z} \sin \left(\delta - \tan^{-1} \frac{R}{X}\right) + E_f^2 \frac{R}{Z^2} \qquad \text{(VI-236)}$$

and

$$\frac{P_r}{3} = \frac{E_f V_b}{Z} \sin \left(\delta + \tan^{-1} \frac{R}{X}\right) - E_f^2 \frac{R}{Z^2} \qquad \text{(VI-237)}$$

where

P_s = sending end power

P_r = receiving end power

E_f, V_b are as defined above as they can be considered as the sending end and receiving end voltage magnitudes

$Z = R + j X$, the series impedance

The difference between P_r and P_s is power dissipated as ohmic loss (heat) in the resistance R.

Let us examine the nature of the double angle, 2δ, part of the expression for power transfer between a "salient pole" alternator (non-cylindrical rotor configuration) and an infinite bus. Recall that inductance, L, is related to reluctance, R, by

$$L = \frac{N\phi}{i} = \frac{N^2\phi}{F} = \frac{N^2}{R} \qquad \text{(VI-238)}$$

From (VI-238)

$$X_q = \omega L_q = N^2\,\omega\left(\frac{1}{R_q}\right) = N^2\,\omega\left(\frac{1}{R_{max}}\right) \qquad \text{(VI-239)}$$

where

$$R_q = \text{quadrature axis reluctance} = R_{max}$$

and

$$X_d = N^2\omega\left(\frac{1}{R_{min}}\right) \qquad \text{(VI-240)}$$

where:

$$R_d = \text{direct axis reluctance} = R_{min}$$

From (VI-239,-240) and the last term in (VI-234)

$$\frac{V_b^2}{2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right) = \frac{V_b^2}{2\omega N^2}\left(R_{max} - R_{min}\right) \sin 2\delta \qquad \text{(VI-241)}$$

Referring back to equation (II-29) for the relationship between voltage, flux, and turns we have (for a device with 2 poles)

$$E_{rms} = \frac{N\,\omega\,\phi_m}{\sqrt{2}} \qquad \text{(VI-242)}$$

Substituting (VI-242) for V_b in the right hand side of (VI-201) yields

$$\left(\frac{N^2\,\omega^2\phi_m^2}{2}\right)\left(\frac{R_{max} - R_{min}}{2\,\omega\,N^2}\right) \sin 2\delta = \frac{\omega\,\phi_m^2}{4}\left(R_{max} - R_{min}\right) \sin 2\delta \qquad \text{(VI-243)}$$

(VI-243) is an expression for power in a specific configuration of machine with a magnetic reluctance variation. Comparison of (VI-243) with the expression derived for torque in a simple reluctance motor indicates that the two expressions are of the same form. Thus, we conclude that a reluctance torque (and a corresponding power) exists in a salient pole synchronous machine. (Obviously, it cannot exist in a cylindrical rotor machine). We refer to both synchronous torque, or power and reluctance torque, or power as contributing to the energy conversion process.

From Figure VI-17 and (VI-234) it is readily seen that for any given condition of load, voltage and excitation there exists a unique power angle, δ , at which the machine will operate. Further, there is some maximum value of power and if power in excess of this maximum capability is demanded of the machine it simply cannot deliver it and loss of synchronous results. For the most simple case, consider the cylindrical rotor machine. Maximum power transfer occurs when $\delta = \pi/2$ radians. If this value is exceeded, loss of synchronism may result. The word "may" is used because the circumstances of how the load is increased are important. If the load is slowly added so that there are no transient excursions of δ, $\pi/2$ would be the limiting value of δ . Thus, $\pi/2$ is referred to as the "steady state" maximum. During transient swings, resulting from impact type loads, δ may exceed $\pi/2$ while the rotor is oscillating toward its new steady state position. Determination of whether synchronism will be lost or maintained must be studied for each specific situation. Such studies are referred to as "stability studies".

The steady state maximum power which can be transferred between source and load if saliency is involved is determined from (VI-234) as follows:

$$\frac{dP}{d\delta} = \frac{E_f V_b}{X_d} \cos \delta_m + V_b^2 \left(\frac{1}{X_q} - \frac{1}{X_d}\right) \cos 2\delta_m = 0 \qquad \text{(VI-244)}$$

where $\delta_m = \delta$ for maximum power transfer.

We can solve for δ_m to satisfy (VI-244) and then substitute it in (VI-234) to obtain P_{max}. Because of the nature of (VI-244) it is necessary to resort to "cut and try", iterative procedures on a computer, or graphical methods.

If the situation under study involved two salient pole machines (one as load, the other as source) (VI-234) is still applicable if we let X_d, X_q be the sum of the values for both machines and V_b be the excitation voltage for the second machine.

VI.9 CHARACTERISTIC CURVES OF THE SYNCHRONOUS MOTOR: The ability to cause a synchronous motor to take power at various power factors determined by the value of field excitation was discussed in Section VI.7. In this section we will derive equations necessary to plot armature current as a function of excitation current for specific values of converted power. We will consider a cylindrical rotor synchronous motor with $X_s >> r$ so that r can be neglected. In effect, the results will relate armature current to field excitation for constant input power. Actual shaft power output would be input power minus losses - notably the i^2r loss and the friction and windage loss.

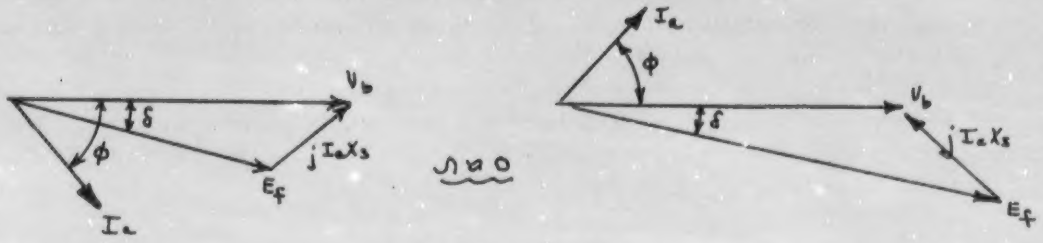
Consider the power input to the motor. It is:

$$P = 3\ V_b I_a \cos \phi \qquad \text{(VI-245)}$$

where P is the three phase power input. It is also given, for the cylindrical rotor machine with negligible armature resistance, by:

$$P = \frac{3V_b E_f}{X_s} \sin \delta \qquad \text{(VI-246)}$$

From the phasor diagram, Figure VI-18:



(a) Lagging p.f. (b) Leading p.f.

Figure VI-18. Cylindrical Rotor Synchronous Motor, $r \approx 0$

$$E_f \cos\delta \pm I_aX_s \sin\phi = V_b \tag{VI-247}$$

or:

$$(I_a \sin\phi)^2 = \left(\frac{V_b - E_f \cos\delta}{X_s}\right)^2 \tag{VI-248}$$

From (VI-245):

$$(I_a \cos\phi)^2 = \left(\frac{P}{3V_b}\right)^2 \tag{VI-249}$$

Adding (VI-248) and (VI-249) and recognizing that $\cos^2\phi + \sin^2\phi = 1$, we have:

$$I_a = \sqrt{\left(\frac{P}{3V_b}\right)^2 + \left(\frac{V_b - E_f \cos\delta}{X_s}\right)^2} \tag{VI-250}$$

where, from (VI-246):

$$\delta = \sin^{-1} \frac{P\,X_s}{3V_b\,E_f} \tag{VI-251}$$

and:

$$E_f = \frac{i_f\,L_{df}\,\omega}{\sqrt{2}} \tag{VI-252}$$

Now, power, P, and terminal voltage V_b, and X_s are specified. We can then calculate E_f for various i_f (from VI-252) substitute in (VI-251) to determine δ. These values can then be substituted in (VI-250) to determine I_a corresponding to fixed, P, V_b, X_s and the chosen value of i_f.

The minimum value of E_f is determined from stability considerations. For any given values of P, X_s, V_b the minimum value of E_f possible is that which results in $\delta = \pi/2$. Thus, from (VI-251,-252) for $\sin \pi/2 = 1$:

$$i_{f\ min} = \frac{E_{f\ min}\sqrt{2}}{\omega L_{df}} = \frac{\sqrt{2}}{\omega L_{df}} \frac{P\ X_s}{3V_b} \tag{VI-253}$$

The maximum value of i_f is determined by the rating of the excitation system. Values of i_f chosen would, then, be over the range

$$i_{f\ min} \leq i_f \leq i_{f\ max} \tag{VI-254}$$

A typical synchronous motor characteristic, called the "Vee Curves" is shown in Figure VI-19:

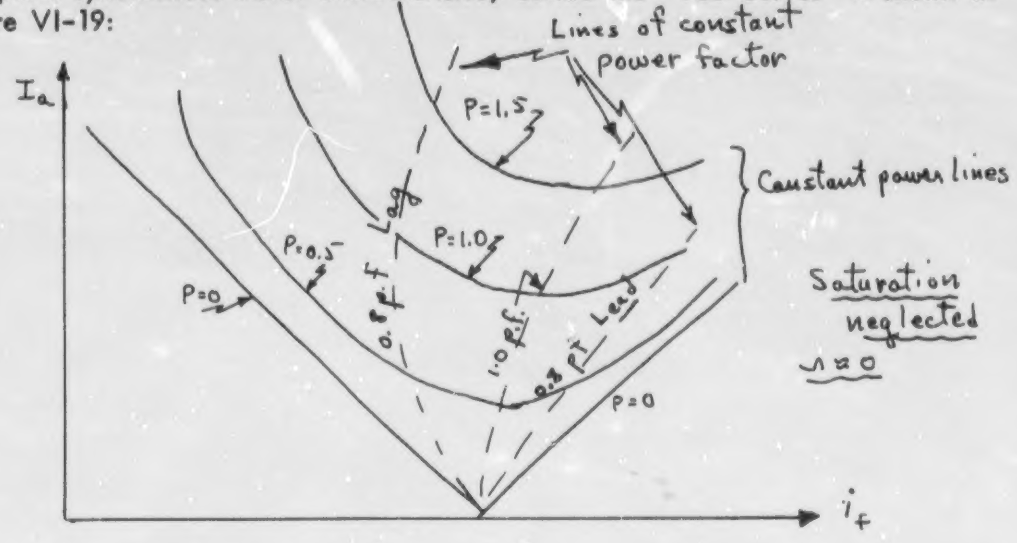


Figure VI-19. Synchronous Motor Vee Curves, Cylindrical Rotor

The current will be a minimum when, for a given power, $I_a \cos \phi = I_a$ and $I_a \sin \phi = 0$. From (VI-248) this occurs when

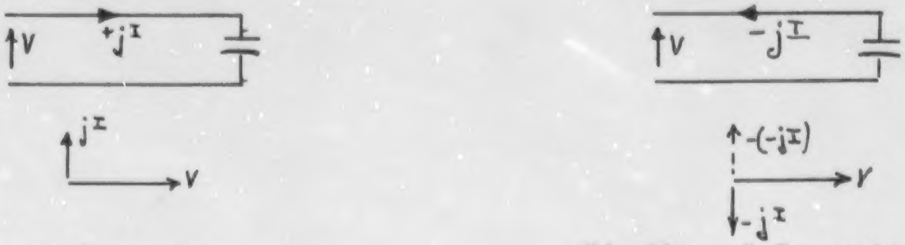
$$\cos \delta = \frac{V_b}{E_f} \tag{VI-255}$$

Dividing (VI-255) into (VI-246) yields (for unity power factor and minimum point on a Vee curve)

$$\delta = \tan^{-1} \frac{X_s P}{3 V_b^2} \qquad \text{(VI-256)}$$

Of course, a curve for P = 0 is not possible in practice because even if no shaft load is present there are finite losses present.

We conclude that the synchronous motor, when under excited absorbs lagging power factor reactive power. When over excited, it absorbs leading power factor reactive power (it appears as a capacitive load to the supply). We can also say that the over excited motor delivers, or supplies, lagging power factor reactive power back into the system. Refer to Figure VI-20. Either convention is correct.



(a) "Load" Convention (b) "Source" Convention

Figure VI-20. "Load" and "Source" Convention for Reactive Power Flows

It should be emphasized that saturation has been neglected in our calculation. In the actual machine, as excitation is increased and the magnetic circuit saturates, it takes a greater increment of current for succeeding equal increments of increasing "excitation" voltage.

To calculate the "Vee curves" for a salient pole motor, refer to the expression for the power angle, in terms of various parameters, derived in a homework assignment, i.e.,

$$\tan \delta = \frac{I_a X_q \cos\phi + I_a r \sin\phi}{V_b + I_a X_q \sin\phi - I_a r \cos\phi} \qquad \text{(VI-257)}$$

Since,

$$\cos\phi = \frac{P}{3 V_b I_a} \qquad \text{(VI-258)}$$

we can determine:

$$\sin \phi = \frac{\sqrt{(3V_b I_a)^2 - P^2}}{3V_b\ I_a} \qquad \text{(VI-259)}$$

For specific V_b and P we can assume a value of I_a. We then:

a) Calculate cos ϕ, sinϕ from (VI-258, -259)

b) Calculate tan δ from (VI-257)

c) From b, find sin δ, sin 2δ

d) Solve (VI-234) for E_f as follows:

$$E_f = \frac{\frac{P}{3} - \frac{{V_b}^2}{2}\left(\frac{1}{X_q} - \frac{1}{X_d}\right) \sin 2\delta}{\frac{V_b}{X_d} \sin \delta}$$

e) Find $i_f = \dfrac{\sqrt{2}\, E_f}{\omega\, L_{df}}$

use values corresponding to:

$$E_{f\ min} \leq E_f \leq E_{f\ max}$$

as discussed previously (after determining $E_{f\ min}$ for stability from the procedures discussed in connection with (VI-204).

In subsequent sections we will analyze transient situations involving synchronous machines.

VI.10 AN ALTERNATOR ON SHORT CIRCUIT: We will study the behavior of an alternator subjected to a short circuit for three reasons:

1) the results will provide insight into physical meaning of some of the defined constants and inductances developed previously
2) the results will enable us to determine various parameters, which are used in analysis, from test results
3) the procedure developed is easily adaptable to the study of transients associated with the switching of loads.

The first situation we will consider is that of a three phase alternator being driven at rated speed. It ia assumed that:

1. the alternator is initially unloaded (an open circuit) and the open circuit voltage is V_{ta} volts, rms, per phase
2. the speed is constant at rated value and remains so during the time of our study
3. all three phases are switched simultaneously at $t = 0$ (therefore no zero sequence quantities exist)
4. the field excitation voltage, E_f, remains constant

If we use the equations (VI-180 through -190) we must remember that the transfer function obtained did not include initial conditions, such as the flux linkages existing at $t = 0$. These equations, since they do not provide for initial values, are strictly valid only for the change, or incremental values, of the variable. The results we obtain are also only changes or incremental values of the variables. Recall it was stated that the incremental values must be added to the initial values to obtain the total variable. Suppose we, in this problem, determine the value of v_d and v_q, at time before switching ($t=0$) when everything is in steady state. If we mathematically introduce $v_d(s) = -\frac{v_d}{s}$ and $v_q(s) = -\frac{v_q}{s}$ at $t = 0$ the total value of the variables are $v_d = 0$, $v_q = 0$ which corresponds to the short circuit condition.

Let us examine the situation prior to $t = 0$. With an open circuit condition, i_d, i_q are zero and the fluxes λ_d and λ_q can be determined as, from (VI-180 thourgh -190):

$$\lambda_d = \frac{L_{df}\, v_f}{r_f} \qquad \text{(VI-260)}$$

$$\lambda_q = 0 \qquad \text{(VI-261)}$$

Now

$$\frac{d\lambda_d}{dt} = 0 \text{ and } \frac{d\lambda_q}{dt} = 0. \quad \text{Therefore;}$$

$$v_d = 0 \qquad \text{(VI-262)}$$

$$v_q = \omega L_{df} \frac{v_f}{r_f} \qquad \text{(VI-263)}$$

From (VI-35), with $v_d = 0$,

$$- v_q \sin \omega t = v_a = \sqrt{2}\, V_{ta} \sin \omega t \qquad \text{(VI-264)}$$

$$\therefore\ v_q = - \sqrt{2}\, V_{ta} \qquad \text{(VI-265)}$$

At $t = 0$, we will insert, mathematically, the negative of this in accordance with our discussion above, i.e.

$$v_d(s) = 0;\ v_q(s) = -\left(\frac{-\sqrt{2}\, V_{ta}}{s}\right) = \frac{\sqrt{2}\, V_{ta}}{s} \qquad \text{(VI-266)}$$

Since we specified that the field excitation voltage is to be held constant, $v_f(s) = 0$. From (VI-180 through -190) we can now write appropriate equations as follows:

$$s\ \lambda_d(s) - \omega \lambda_q(s) + r\ i_d(s) = 0 \qquad \text{(VI-267)}$$

$$\omega \lambda_d(s) + s\ \lambda_q(s) + r\ i_q(s) = \frac{\sqrt{2}\, V_{ta}}{s} \qquad \text{(VI-268)}$$

where:

$$\lambda_d(s) = X_2(s)\ i_d(s) \text{ and } \lambda_q(s) = X_1(s)\ i_q(s) \qquad \text{(VI-269)}$$

Substituting (VI-269) into (VI-267,-268) yields:

$$[r + s\ X_2(s)]\ i_d(s) - \omega X_1(s)\ i_q(s) = 0 \qquad \text{(VI-270)}$$

$$\omega\ X_2(s)\ i_d(s) + [r + s\ X_1(s)]\ i_q(s) = \frac{\sqrt{2}\ V}{s} \qquad \text{(VI-271)}$$

Solving for $i_d(s)$, $i_q(s)$;

$$i_d(s) = \frac{\begin{vmatrix} 0 & - \omega X_1(s) \\ \dfrac{\sqrt{2}\, V_{ta}}{s} & r + s\ X_1(s) \end{vmatrix}}{\begin{vmatrix} r + s\ X_2(s) & - \omega X_1(s) \\ \omega X_2(s) & r + s\ X_1(s) \end{vmatrix}} \qquad \text{(VI-272)}$$

$$i_q(s) = \frac{\begin{vmatrix} r + s\,X_2(s) & 0 \\ \omega\,X_2(s) & \dfrac{\sqrt{2}\,V_{ta}}{s} \end{vmatrix}}{\begin{vmatrix} r + s\,X_2(s) & -\omega\,X_1(s) \\ \omega\,X_2(s) & r + s\,X_1(s) \end{vmatrix}} \tag{VI-273}$$

From (VI-272)

$$i_d(s) = \frac{\dfrac{\sqrt{2}\,V_{ta}}{s}\, X_1(s)\,\omega}{r^2 + (r\,X_1(s) + r\,X_2(s))s + s^2\,X_1(s)X_2(s) + X_1(s)\,X_2(s)\,\omega^2} \tag{VI-274}$$

$$= \frac{\sqrt{2}\,V_{ta}\omega}{X_2(s)}\;\frac{1}{s}\left\{\frac{1}{s^2 + \left(\dfrac{1}{X_1(s)} + \dfrac{1}{X_2(s)}\right)r\;s + \omega^2 + \dfrac{r^2}{X_1(s)\;X_2(s)}}\right\} \tag{VI-275}$$

In (VI-275) we note that r is usually (in a well designed machine) relatively small on a per unit basis. We will neglect the r^2 term completely. Thus,

$$\omega^2 + \frac{r^2}{X_1(s)\;X_2(s)} \approx \omega^2 \tag{VI-276}$$

Further, if r_f, r_{kd}, and r_{kq} are small, we can greatly simplify the coefficient of s in the denominator of (VI-275) because, with this assumption:

$$\begin{array}{ll} 1 + T'_{do}\,s \rightarrow T'_{do}\,s & 1 + T''_{do}\,s \rightarrow T''_{do}\,s \\ 1 + T'_{d}\,s \rightarrow T'_{d}\,s & 1 + T''_{d}\,s \rightarrow T''_{d}\,s \\ 1 + T''_{qo}\,s \rightarrow T''_{qo}\,s & 1 + T''_{q}\,s \rightarrow T''_{q}\,s \end{array} \tag{VI-277}$$

then,

$$\frac{1}{X_1(s)} + \frac{1}{X_2(s)} \approx \frac{T''_{qo}\ s}{L_q T''_q\ s} + \frac{(T'_{do}\ s)(T''_{do}\ s)}{L_d (T'_d\ s)(T''_d\ s)} \qquad \text{(VI-278)}$$

From (VI-178):

$$\frac{T'_{do}\ T''_{do}}{L_d T'_d\ T''_d} = \frac{1}{L''_d} \qquad \text{(VI-279)}$$

From (VI-279);

$$\frac{T''_{qo}}{L_q\ T''_q} = \frac{1}{L''_q} \qquad \text{(VI-280)}$$

and;

$$\frac{1}{X_1(s)} + \frac{1}{X_2(s)} \approx \left(\frac{1}{L''_q} + \frac{1}{L''_d}\right) \qquad \text{(VI-281)}$$

Using the value of $X_2(s)$ and the approximations (VI-281 and -276) in (VI-275) yields:

$$i_d(s) = \frac{(\sqrt{2})\,\omega\, V_{ta}}{L_d} \left\{ \frac{(1 + T'_{do}\ s)(1 + T''_{do}\ s)}{s\ (1 + T'_d\ s)(1 + T''_d\ s)(\ s^2 + 2as + \omega^2)} \right\} \qquad \text{(VI-282)}$$

where:

$$a = \frac{r}{2}\left(\frac{1}{L''_d} + \frac{1}{L''_q}\right) \qquad \text{(VI-283)}$$

In order to obtain $i_d(t)$, several assumptions will be made. Past experience indicates that the results obtained, after simplifying assumptions are made, agree very closely with experimental results. We will first examine the term:

$$s^2 + 2as + \omega^2 \qquad \text{(VI-284)}$$

This can be factored into:

$$s = -\ a \pm j\sqrt{\omega^2 - a^2} \qquad \text{(VI-285)}$$

Since r is a relatively small number, a is also small, ω is relatively large, and:

$$a^2 << \omega^2 \qquad \text{(VI-286)}$$

from which:

$$s \approx - a \pm j\omega \qquad \text{(VI-287)}$$

or:

$$s = - \alpha_1, \; s = - \alpha_2 \qquad \text{(VI-288)}$$

where

$$-\alpha_1 \approx - a + j\omega ; \; - \alpha_2 \approx - a - j\omega \qquad \text{(VI-289)}$$

The portion of (VI-282) in brackets can be written as: (using partial fraction expansion techniques):

$$\frac{(1 + T'_{do}s)(1 + T''_{do}\ s)}{s(1 + T'_d\ s)(1 + T''_d\ s)(s + \alpha_1)(s + \alpha_2)} = \frac{K_1}{s} + \frac{K_2}{1 + T'_d s} +$$

$$\frac{K_3}{1 + T''_d s} + \frac{K_4/\alpha_1}{\frac{s}{\alpha_1} + 1} + \frac{K_5/\alpha_2}{\frac{s}{\alpha_2} + 1} \qquad \text{(VI-290)}$$

The coefficients in (VI-290) can be evaluated in the conventional fashion as:

$$K_1 = \frac{(1 + T'_{do}\ s)(1 + T''_{do}\ s)}{(1 + T'_d\ s)(1 + T''_d\ s)} \; \frac{1}{(s^2 + 2as + \omega^2)} \Bigg|_{s=0} = \frac{1}{\omega^2} \qquad \text{(VI-291)}$$

$$K_2 = \frac{(1 + T'_{do}\ s)(1 + T''_{do}\ s)}{s(1 + T''_d\ s)(\omega^2 + 2as + s^2)} \Bigg|_{s = -\frac{1}{T'_d}} = \frac{(1 - \frac{T'_{do}}{T'_d})(1 - \frac{T''_{do}}{T'_d})(- T'_d)}{(1 - \frac{T''_d}{T'_d})\left[\frac{1}{(T'_d)^2} - \frac{2a}{T'_d} + \omega^2)\right]} \qquad \text{(VI-292)}$$

Now, from our previous discussion, we know that:

$$T'_d >> T''_{do} \; ; \; T'_d >> T''_d \qquad \text{(VI-293)}$$

because the transient phenomena time constants are associated with the field circuit (F) whereas the subtransient are determined from the damper circuits, KD and KQ. T'_d is on the order of several seconds and "a" is less than one. Therefore, we can also make the approximation that

$$\left(\frac{1}{{T'_d}^2} - \frac{2a}{T'_d} + \omega^2\right) \approx \omega^2 \qquad \text{(VI-291)}$$

and (VI-292) becomes:

$$K_2 \approx \left(\frac{T'_{do} - T'_d}{\omega^2}\right) \qquad \text{(VI-295)}$$

Similarly,

$$K_3 = \left.\frac{(1 + T'_{do}\, s)(1 + T''_{do}\, s)}{s(1 + T'_d\, s)(s^2 + 2as + \omega^2)}\right|_{s = -\frac{1}{T''_d}} = \frac{-(T''_d)\left(1 - \frac{T'_{do}}{T''_d}\right)\left(1 - \frac{T''_{do}}{T''_d}\right)}{\left(1 - \frac{T'_{do}}{T''_d}\right)\left(\frac{1}{{T''_d}^2} - \frac{2a}{T''_d} + \omega^2\right)} \qquad \text{(VI-296)}$$

Using the same assumptions as above:

$$K_3 \approx \frac{T'_{do}}{T'_d} \; \frac{(T''_{do} - T''_d)}{\omega^2} \qquad \text{(VI-297)}$$

Similarly:

$$K_4 = \left.\frac{(1 + T'_{do}\, s)(1 + T''_{do}\, s)}{s(1 + T'_d\, s)(1 + T''_d\, s)(s + \alpha_2)}\right|_{s = -\alpha_1} = \frac{(1 - \alpha_1 T'_{do})(1 - \alpha_1 T''_{do})}{-(\alpha_1)(1 - T'_d \alpha_1)(1 - \alpha_1 T''_d)(-\alpha_1 + \alpha_2)} \qquad \text{(VI-298)}$$

If $\alpha_1 T'_{do}$, $\alpha_1 T''_{do}$, $\alpha_1 T'_d$, $\alpha_1 T''_d$ are all considerably larger than 1,

$$K_4 \approx \frac{T'_{do}\, T''_{do}}{T'_d\, T''_d} \; \alpha_1(\alpha_1 - \alpha_2) \qquad \text{(VI-299)}$$

This is a reasonable assumption because α_1, α_2 in magnitude are greater than the magnitude ω.

Using the values of α_1, α_2 from (VI-287, -288)

$$\alpha_1 = a - j\omega \; ; \; \alpha_2 = a + j\omega \tag{VI-300}$$

or

$$\alpha_1 \approx -j\omega \; ; \; \alpha_2 \approx +j\omega \tag{VI-301}$$

(VI-299) becomes

$$K_4 \approx -\frac{T'_{do}\, T''_{do}}{T'_d T''_d \;\, 2\,\omega^2} \tag{VI-302}$$

From (VI-279)

$$\frac{T'_{do}\, T''_{do}}{T'_d\, T''_d} = \frac{L_d}{L''_d} \tag{VI-303}$$

and we can write (VI-302) as, using (VI-303)

$$K_4 \approx -\frac{L_d}{L''_d}\, \omega^2 \left(\frac{1}{2}\right) \tag{VI-304}$$

K_5 can be determined as, using the same approximation

$$K_5 \approx \frac{T'_{do}\, T''_{do}}{T'_d\, T''_d\, \alpha_2(\alpha_2 - \alpha_1)} \approx -\frac{L_d}{L''_d\, \omega^2} \left(\frac{1}{2}\right) \tag{VI-305}$$

Using the expression for K_1, K_2, K_3, K_4 and K_5 as determined above in the partial fraction expansion (VI-290) and (VI-282) we find the approximate expression for $i_d(s)$ as:

$$i_d(s) = \frac{\sqrt{2}\, V_{ta}}{\omega\, L_d} \left\{ \frac{1}{s} + \frac{(T'_{do} - T'_d)}{(1 + T'_d\, s)} + \frac{T'_{do}}{T'_d}\, \frac{(T''_{do} - T''_d)}{(1 + T''_d\, s)} + \right.$$
$$\left. - \frac{L_d}{L''_d} \left(\frac{1}{2}\right) \left[\frac{1/\alpha_1}{s/\alpha_1 + 1} + \frac{1/\alpha_2}{s/\alpha_2 + 1} \right] \right\} \tag{VI-306}$$

Note that:

$$\mathcal{L}^{-1}\left[\frac{K}{s\,T+1}\right] = \frac{K}{T}\,\varepsilon^{-t/T} \qquad \text{(VI-307)}$$

The inverse transform of the two terms containing α in (VI-306) becomes:

$$-\frac{L_d}{L''_d}\left[\frac{\varepsilon^{-\alpha_1 t} + \varepsilon^{-\alpha_2 t}}{2}\right] = -\frac{L_d}{L''_d}\,\varepsilon^{-at}\left[\frac{\varepsilon^{j\omega t} + \varepsilon^{-j\omega t}}{2}\right] = -\frac{L_d}{L''_d}\,\varepsilon^{-at}\cos\omega t \qquad \text{(VI-308)}$$

We can define a time constant;

$$\frac{1}{T_a} = a \qquad \text{(VI-309)}$$

where a has the value from (VI-283), i.e.,

$$T_a = \frac{2}{r}\left(\frac{L''_d\,L''_q}{L''_d + L''_q}\right) \qquad \text{(VI-310)}$$

We can now write the inverse transform of (VI-306) as:

$$i_d(t) = \frac{\sqrt{2}\,V_{ta}}{\omega\,L_d}\left\{1 + \frac{(T'_{do} - T'_d)}{T'_d}\,\varepsilon^{-t/T'_d} + \frac{(T''_{do} - T''_d)}{T'_d\,T''_d}\,T'_{do}\,\varepsilon^{-t/T''_d} - \frac{L_d}{L''_d}\,\varepsilon^{-t/T_a}\cos\omega t\right\} \qquad \text{(VI-311)}$$

Rearranging (VI-311) yields:

$$i_d(t) = \sqrt{2}\,V_{ta}\left[\frac{1}{\omega L_d} + \left(\frac{T'_{do}}{\omega L_d T'_d} - \frac{1}{\omega L_d}\right)\varepsilon^{-t/T'_d} + \left(\frac{1}{\omega L_d}\,\frac{T'_{do}\,T''_{do}}{T'_d\,T''_d} - \frac{T'_{do}}{\omega L_d T'_d}\right)\varepsilon^{-t/T''_d} - \frac{1}{\omega L''_d}\,\varepsilon^{-t/T_a}\cos\omega t\right] \qquad \text{(VI-312)}$$

From (VI-177):

$$\frac{T'_{do}}{\omega L_d\, T'_d} = \frac{1}{\omega L'_d} \qquad \text{(VI-313)}$$

From (VI-178)

$$\frac{T'_{do}\; T''_{do}}{\omega L_d\, T'_d\, T''_d} \rightarrow \frac{1}{\omega L''_d} \qquad \text{(VI-314)}$$

We will denote the following reactances

ωL_d = direct axis synchronous reactance = X_d (VI-315)

$\omega L'_d$ = direct axis transient reactance = X'_d (VI-316)

$\omega L''_d$ = direct axis subtransient reactance = X''_d (VI-317)

With these definitions, we can express the current in (VI-312) as a function of voltage and impedance. Thus:

$$i_d(t) = \frac{\sqrt{2}\, V_{ta}}{X_d} + \sqrt{2}\, V_{ta} \left(\frac{1}{X'_d} - \frac{1}{X_d}\right) \varepsilon^{-t/T'_d} +$$

$$+ \sqrt{2}\, V_{ta} \left(\frac{1}{X''_d} - \frac{1}{X'_d}\right) \varepsilon^{-t/T''_d} - \frac{\sqrt{2}\, V_{ta}}{X''_d}\, \varepsilon^{-t/T_a} \cos \omega t \qquad \text{(VI-318)}$$

We will now find $i_q(t)$ from (VI-273):

$$i_q(s) = \frac{\sqrt{2}\, V_{ta}\, (r + s\, X_2(s))}{s \left\{ r^2 + r(X_1(s) + X_2(s))s + X_1(s)\, X_2(s)\, s^2 + \omega^2 X_1(s)\, X_2(s) \right\}} \qquad \text{(VI-319)}$$

$$\approx \frac{\sqrt{2}\, V_{ta}}{X_1(s)} \left\{ \frac{1}{s^2 + r\left(\frac{1}{X_1(s)} + \frac{1}{X_2(s)}\right)s + \omega^2} \right\} \qquad \text{(VI-320)}$$

$$\approx \frac{\sqrt{2}\, V_{ta}}{L_q} \left\{ \frac{(1 + T''_{qo}\, s)}{(1 + T''_q\, s)(s^2 + 2as + \omega^2} \right\} \qquad \text{(VI-321)}$$

using the value of $X_1(s)$ from (VI-186), the value of a from (VI-283) and the same assumptions used in simplifyng the expression for $i_d(s)$.

Again, using partial fraction expansion

$$\frac{(1 + T''_{qo}\ s)}{(1 + T''_q\ s)(s^2 + 2as + \omega^2)} \approx \frac{K_6}{(1 + T''_q s)} + \frac{K_7/\alpha_1}{s/\alpha_1 + 1} + \frac{K_8/\alpha_2}{s/\alpha_2 + 1} \qquad \text{(VI-322)}$$

where α_1, α_2 have the values as defined in (VI-289):

$$K_6 = \left.\frac{(1 + T''_{qo}\ s)}{s^2 + 2a + \omega^2)}\right|_{s = -\frac{1}{T''_q}} \approx \frac{T''_q - T''_{qo}}{T''_{qo}\ \omega^2} \qquad \text{(VI-323)}$$

using the same arguments, as before, regarding relative values of time constants.

For a large number of machines, the coupling between the KQ and Q windings is rather poor because of the high reluctance of the path through the air gap along the q axis and:

$$T''_q \approx T''_{qo} \qquad \text{(VI-324)}$$

which results in:

$$K_6 = 0 \qquad \text{(VI-325)}$$

$$K_7 = \left.\frac{(1 + T''_{qo}\ s}{(1 + T''_q s)(s + \alpha_2)}\right|_{s = -\alpha_1} = \frac{(1 - \alpha_1\ T''_{qo})}{(1 - \alpha_1 T''_q)(\alpha_2 - \alpha_1)} \qquad \text{(VI-326)}$$

from which, with previous arguments and assumption:

$$K_7 \approx \frac{T''_{qo}}{T''_q\ (2j\omega)} \qquad \text{(VI-327)}$$

Similarly:

$$K_8 = \left.\frac{(1 + T''_{qo}\ s)}{(1 + T''_q\ s)(s + \alpha_1)}\right|_{s = -\alpha_2} = \frac{1 - \alpha_2 T''_{qo}}{(1 - T''_q\ \alpha_2)(\alpha_1 - \alpha_2)} \qquad \text{(VI-328)}$$

or

$$K_8 \approx - \frac{T''_{qo}}{T''_q\ (2j\omega)} \qquad \text{(VI-329)}$$

Using the values of K_6, K_7, and K_8 in (VI-322) and (VI-321) yields:

$$i_q(s) = \frac{\sqrt{2}\ V_{ta}}{L_q}\ \frac{T''_{qo}}{T''_q}\ \left(\frac{1}{2j\omega}\right)\left(\frac{1/\alpha_1}{s/\alpha_1 + 1} + \frac{1/\alpha_2}{s/\alpha_2 + 1}\right) \qquad \text{(VI-330)}$$

From (VI-179):

$$\omega \frac{T''_{qo}}{L_q\ T''_q} = \frac{1}{\omega L''_q} = \frac{1}{X''_q} \qquad \text{(VI-331)}$$

where X''_q = quadrature axis subtransient reactance.

Using (VI-331) in (VI-330) and taking the inverse transform yields:

$$i_q(t) = \frac{\sqrt{2}\ V_{ta}}{X''_q}\ \varepsilon^{-at}\ \left(\frac{\varepsilon^{j\omega t} - \varepsilon^{-j\omega t}}{2j}\right) \qquad \text{(VI-332)}$$

$$= + \frac{\sqrt{2}\ V_{ta}}{X''_q}\ \varepsilon^{-t/T_a}\ \sin \omega t \qquad \text{(VI-333)}$$

For balanced conditions, i.e., no zero sequence quantities, from (VI-35):

$$i_a = i_d \cos \theta - i_q \sin \theta \qquad \text{(VI-334)}$$

where $\theta = \omega t + \beta$ and β is an arbitrary phase angle.

From (VI-318):

$$i_d = A - B \cos \omega t$$

where

$$A = \frac{\sqrt{2}\, V_{ta}}{X_d} + \sqrt{2}\, V_{ta} \left(\frac{1}{X'_d} - \frac{1}{X_d}\right) \varepsilon^{-t/T'_d} + \sqrt{2}\, V_{ta} \left(\frac{1}{X''_d} - \frac{1}{X'_d}\right) \varepsilon^{-t/t''_d} \qquad \text{(VI-335)}$$

$$B = \frac{\sqrt{2}\, V_{ta}}{X''_d} \varepsilon^{-t/T_a} \qquad \text{(VI-336)}$$

From (VI-333):

$$i_q = C \sin \omega t \qquad \text{(VI-337)}$$

where:

$$C = \frac{\sqrt{2}\, V_{ta}}{X''_q} \varepsilon^{-t/T_a} \qquad \text{(VI-338)}$$

Then

$$i_a(t) = (A - B \cos \omega t) \cos (\omega t + \beta) - C \sin \omega t \sin(\omega t + \beta) \qquad \text{(VI-339)}$$

$$i_a(t) = A \cos (\omega t + \beta) - [B \cos \omega t \cos (\omega t + \beta) + C \sin \omega t \sin (\omega t + \beta)] \qquad \text{(VI-340)}$$

Now:

$$B \cos \omega t \cos (\omega t + \beta) = \frac{B}{2} [\cos \beta + \cos (2\omega t + \beta)] \qquad \text{(VI-341)}$$

$$C \sin \omega t \sin \omega t + \beta = \frac{C}{2} [\cos \beta - \cos (2\omega t + \beta)] \qquad \text{(VI-342)}$$

(VI-340) becomes:

$$i_a(t) = A \cos (\omega t + \beta) - \left(\frac{B+C}{2}\right) \cos \beta - \left(\frac{B-C}{2}\right) \cos(2\,\omega t + \beta) \qquad \text{(VI-343)}$$

Using the values of A, B, and C from (VI-335, -336, and -338) we have:

$$i_a(t) = \sqrt{2}\, V_{ta}\left\{\frac{1}{X_d} + \left(\frac{1}{X'_d} - \frac{1}{X_d}\right) \varepsilon^{-t/T'_d} + \left(\frac{1}{X''_d} - \frac{1}{X'_d}\right) \varepsilon^{-t/T''_d}\right\} \cos (\omega t + \beta)$$

$$- \sqrt{2}\, V_{ta} \left(\frac{X''_q + X''_d}{2X''_q X''_d}\right) \cos \beta - \sqrt{2}\, V_{ta} \left(\frac{X''_q - X''_d}{2X''_q X''_d}\right) \cos (2\omega t + \beta) \qquad \text{(VI-344)}$$

The values of i_b and i_c would be of the same form as the expression for i_a except we would replace β by $(\beta - \frac{2\pi}{3})$ and $(\beta - \frac{4\pi}{3})$ respectively.

We note from (VI-344) that the short circuit current consists of three distinct components - a component at synchronous frequency, a unidirectional component and a doulbe frequency component. The fundamental frequency component has an initial value of:

$$\frac{\sqrt{2}\, V_{ta} \cos \beta}{X''_d} \qquad \text{(VI-345)}$$

and it decays at a rate determined by the transient and subtransient time constants. The subtransient time constant is primarily determined by the damper bars. The transient time constant is most heavily influenced by the field circuit and is considerably longer than the subtransient time constant. The fundamental frequency component has a "steady state" value of $\frac{V_{ta}}{X_d}$ ampere rms. The unidirection component magnitude is determined by the exact position of the rotor at the instant of short circuit. If $\beta = \pi$, it is a maximum and i_a is said to be "fully offset", i.e., it is fully asymmetrical about the zero current axis. If the rotor position at the instant of short circuit is such that $\beta = \pi/2, \pi/2 + 2\pi/3$, or $\pi/2 + 4\pi/3$, one of the phase currents will not have a unidirectional component and will have "zero offset" or it can be referred to as "symmetrical".

The second harmonic variation arises because of the "saliency" of the rotor. If the rotor were "cylindrical", i.e., the reluctance is the same in the q and d axis, $(x''_q - x''_d)$ is zero and no second harmonic of current exists. Figure VI-21 shows oscillograms of the three phase currents. Phase "a" current is symmetrical - the other two phases have offset. Figure VI-22 shows the envelope of the steady state, the transient, and the subtransient portion of the fundamental component of the short circuit current.

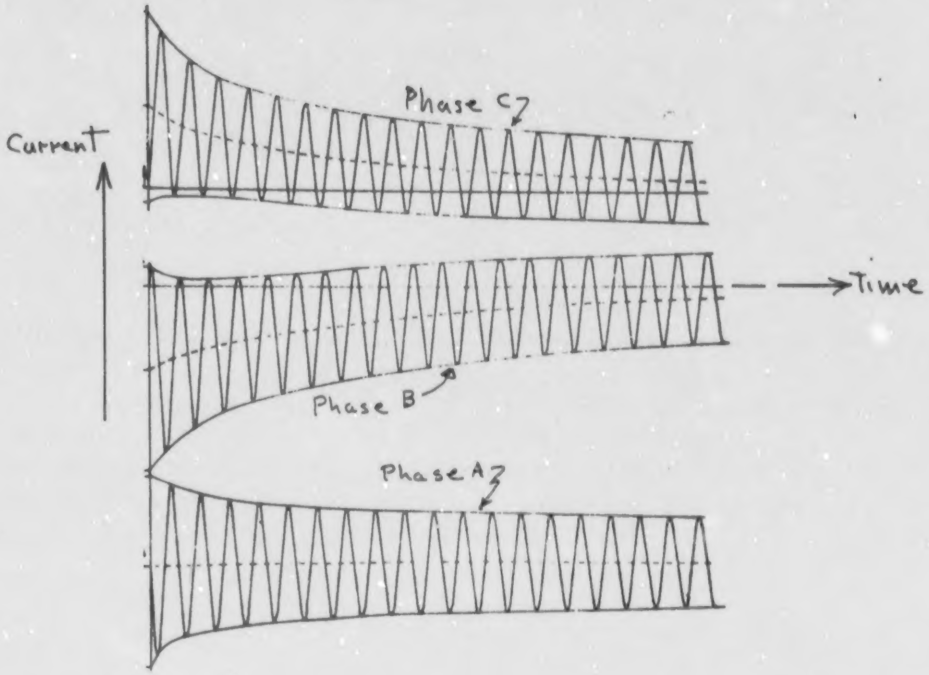


Figure VI-21. Typical Alternator Phase Currents on Short Circuit

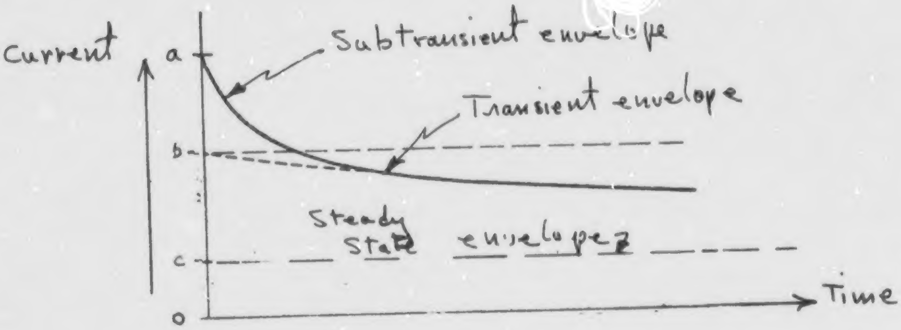


Figure VI-22. Short Circuit Component Envelopes

It is instructive to look at the "worst" case inso far as magnitude is concerned. The magnitude is important because of the mechanical forces on the winding which result and because protective devices must be capable of interrupting the current. Note that, from (VI-318), $i_a(0)$ is zero. This is in accordance with the initial conditions chosen, i.e., no load prior to $t = 0$. If we assume a cylindrical rotor the second harmonic term is zero - in a salient pole machine it is much less than the unidirectional component under any circumstance. Therefore we will neglect it - which is also a way of saying that $x''_d \approx x''_q$. If we are interested in the maximum value of i_a, which can occur within 1/2 cycle, we can also assume no decay of the transient portion. This assumption is conservative because the actual decay results in a lesser value of i_a than that calculated without decay. Then:

$$i_a(t) = \frac{\sqrt{2}\, V_{ta}}{X''_d} \left[\cos(\omega t + \beta) - \cos\beta\right] \qquad \text{(VI-346)}$$

Obviously, for $\beta = \pi$ and $(\omega t + \beta) = 2\pi$, $i_a(t)$ has its maximum value, which is:

$$i_a(t)_{max} = \frac{2\sqrt{2}\, V_{ta}}{X''_d} \qquad \text{(VI-347)}$$

Thus, as an approximation we can say that a full offset wave, i.e., for the situation where $\beta = \pi$ the resulting maximum value of current is two times the maximum value for a symmetrical wave, $\beta = \frac{\pi}{2}$ and occurs at time $t = \frac{\pi}{\omega}$ seconds after switching. This is 1/2 cycle after $t = 0$. Also, the maximum value is determined from the ratio of excitation voltage to subtransient reactance for the condition of no load prior to the short circuit.

Because of the changing fluxes in the machine, voltages will be induced and currents will flow in field circuit winding. In order to evaluate these, refer to the appropriate equation.

At $t = 0$, with $i_d = 0$, $i_q = 0$, $V_q = \sqrt{2}\, V_{ta}$ and at rated speed, using (VI-263):

$$v_q = \omega \lambda_d \quad \text{or} \quad \lambda_d = \frac{-\sqrt{2}\, V_{ta}}{\omega} \qquad \text{(VI-348)}$$

and, since:

$$\lambda_d = \frac{L_{df}}{r_f} V_f = L_{df}\, I_f(0) \qquad \text{(VI-349)}$$

$$I_f(0) = \frac{-\sqrt{2}\, V_{ta}}{\omega\, L_{df}} \tag{VI-350}$$

In the same fashion as before, we will determine the transient portion of i_f and add it to the steady state value $I_f(0)$ existing at $t = 0$.

From the transformed equations (VI-129) through (VI-133), we can substitute (VI-130) for λ_f into (VI-132) yielding: (Since V_f = constant, the change in excitation voltage is zero).

$$r_f(1 + \frac{L_f s}{r_f})\, i_f(s) + s\, L_{df}\, i_{kd}(s) = -\, s\, L_{df}\, i_d(s) \tag{VI-351}$$

Substituting (VI-131) for λ_{kd} into (VI-133) yields

$$s\, L_{df}\, i_f(s) + r_{kd}\, (1 + \frac{L_{kd}}{r_{kd}})\, i_{kd}(s) = -\, s\, L_{df}\, i_d(s) \tag{VI-352}$$

Solving for $i_f(s)$, we have:

$$i_f(s) = \frac{\begin{vmatrix} -\, s\, L_{df}\, i_d(s) & r_{kd}\, (1 + \frac{L_{kd}}{r_{kd}} s) \\ -\, s\, L_{df}\, i_d(s) & s\, L_{df} \end{vmatrix}}{\begin{vmatrix} s\, L_{df} & r_{kd}\, (1 + \frac{L_{kd}}{r_{kd}} s) \\ r_f\, (1 + \frac{L_f s}{r_f}) & s\, L_{df} \end{vmatrix}} \tag{VI-353}$$

$$i_f(s) = \frac{L_{df}\, r_{kd}\, [1 + \frac{L_{kd} - L_{df}}{r_{kd}}\, s]\, s\, i_d(s)}{r_f\, r_{kd}\, [1 + (\frac{L_{kd}}{r_{kd}} + \frac{L_f}{r_f})\, s + (\frac{L_{kd}\, L_f - L^2_{df}}{r_f\, r_{kd}})\, s^2]} \tag{VI-358}$$

Now,

$$\frac{L_{kd} - L_{df}}{r_{kd}} = \frac{\ell_{kd}}{r_k} = T_{kd} \tag{VI-359}$$

as previously defined.

Note the bracketed portion of (VI-358), in the denominator, is the same as the coefficient of $\lambda_d(s)$ in (VI-139). With the approximations used there and using the same defined time constants, (VI-358) becomes:

$$i_f(s) = \frac{L_{df}}{r_f} \frac{[1 + T_{kd}\, s]\, s\, i_d(s)}{[(1 + T'_{do}s)(1 + T''_{do}s)]} \qquad \text{(VI-360)}$$

Using $i_d(s)$ from (VI-282) in (VI-360) yields:

$$i_f(s) = \frac{\sqrt{2}\,\omega V_{ta}}{L_d} \left(\frac{L_{df}}{r_f}\right) \left[\frac{(1 + T_{kd}\, s)}{(1 + T'_d\, s)(1 + T''_d\, s)(s^2 + 2as + \omega^2)}\right] \qquad \text{(VI-361)}$$

Again with partial fraction expansion,

$$\frac{1 + T_{kd}\, s}{(1 + T'_d\, s)(1 + T''_d\, s)(s^2 + 2as + \omega^2)} = \frac{K_1}{(1+T'_d s)} + \frac{K_2}{1+T''_d s} + \frac{K_3/\alpha_1}{s/\alpha_1+1} + \frac{K_4/\alpha_2}{s/\alpha_2 + 1} \qquad \text{(VI-362)}$$

where:

$$\alpha_1 = a - j\omega \qquad \text{(VI-363)}$$

$$\alpha_2 = a + j\omega \qquad \text{(VI-364)}$$

$$K_1 = \left.\frac{1 + T_{kd}\, s}{(1 + T''_d\, s)(s^2 + 2as + \omega^2)}\right|_{s = -\frac{1}{T'_d}} \approx \frac{1}{\omega^2} \qquad \text{(VI-365)}$$

$$K_2 = \left. \frac{(1 + T_{kd}\ s)}{(1 + T'_d\ s)(s^2 + 2as + \omega^2)} \right|_{s = -\frac{1}{T''_d}} \approx - \frac{T''_d}{T'_d \omega^2} \left(1 - \frac{T_{kd}}{T''_d}\right) \tag{VI-366}$$

$$K_3 = \left. \frac{(1 + T_{kd}\ s)}{(1 + T'_d s)(1 + T''_d s)(s + \alpha_2)} \right|_{s = -\alpha_1} \approx \frac{- \alpha_1 T_{kd}}{(- T'_d \alpha_1)(-\alpha_1 T''_d)(\alpha_2 - \alpha_1)} \tag{VI-367}$$

$$\approx - \frac{T_{kd}}{T'_d\ T''_d\ 2\omega^2} \tag{VI-368}$$

$$K_4 = \left. \frac{(1 + T_{kd}\ s)}{(1 + T'_d\ s)(1 + T''_d\ s)(s + \alpha_1)} \right|_{s = -\alpha_2} \approx \frac{- T_{kd}}{T'_d\ T''_d\ 2\omega^2} \tag{VI-369}$$

and

$$i_f(s) = \frac{\sqrt{2}\ V_{ta}}{\omega L_d} \left(\frac{L_{df}}{r_f}\right) \left\{ \frac{1}{(1 + T'_d\ s)} - \frac{T''_d}{T'_d} \left(1 - \frac{T_{kd}}{T''_d}\right) \frac{1}{1 + T''_d\ s} + \right.$$

$$\left. - \frac{T_{kd}}{T'_d\ T''_d} \left(\frac{1/\alpha_1}{s/\alpha_1 + 1} + \frac{1/\alpha_2}{s/\alpha_2 + 1} \right) \right\} \tag{VI-370}$$

From (VI-350)

$$\frac{\sqrt{2}\ V_{ta}}{\omega\ L_{df}} = - I_f(0) \tag{VI-371}$$

Using (VI-371) in (VI-370) and taking the inverse transform yields:

$$i_f(t) = - \frac{I_f(0)}{T'_d} \frac{L_{df}^2}{r_f L_d} \left\{ \varepsilon^{-t/T'_d} - \left(1 - \frac{T_{kd}}{T''_d}\right) \varepsilon^{-t/T''_d} + \right.$$

$$\left. - \frac{T_{kd}}{T''_d} \varepsilon^{-t/T_a} \cos \omega t \right\} \qquad \text{(VI-372)}$$

Now:

$$\frac{L_{df}^2}{r_f L_d T'_d} = \frac{1}{r_f T'_d} (L_d - \ell_a) \frac{L_{df}}{L_d} \qquad \text{(VI-373)}$$

$$= \frac{1}{r_f T'_d} \left(1 - \frac{\ell_a}{L_d}\right) L_{df} + \ell_f - \ell_f) \qquad \text{(VI-374)}$$

$$= \frac{1}{T'_d r_f} \left\{ L_{df} + \ell_f - \left(\ell_f + \frac{\ell_a L_{df}}{\ell_a + L_{df}}\right) \right\} \qquad \text{(VI-375)}$$

From the definitions:

$$T'_{do} = \frac{L_{df} + \ell_f}{r_f} = \text{direct axis transient open circuit time constant}$$

$$T'_d = \quad {}_f = \frac{L_{df} \ell_a}{L_{df} + \ell_a} = \text{direct axis transient short circuit time constant}$$

(VI-375) becomes:

$$\frac{1}{T'_d} (T'_{do} - T'_d) \qquad \text{(VI-376)}$$

The change in field current (VI-372) becomes then,

$$i_f(t) = - I_f(0) \frac{(T'_{do} - T'_d)}{T'_d} \left\{ \varepsilon^{-t/T'_d} - (1 - \frac{T_{kd}}{T''_d}) \varepsilon^{-t/T''_d} + \right.$$

$$\left. - \frac{T_{kd}}{T''_d} \varepsilon^{-t/T_a} \cos \omega t \right\} \qquad \text{(VI-377)}$$

From (VI-177),

$$\frac{L'_d}{L_d} = \frac{T'_d}{T'_{do}} \quad \text{or} \quad \frac{T'_{do} - T'_d}{T'_d} = \frac{L_d - L'_d}{L'_d} \qquad \text{(VI-378)}$$

(VI-377) is the transient portion of the field current. This can be added to the steady state portion, $I_f(0)$, to obtain total field current as a function of time after the alternator, at no load and driven at constant speed, is subjected to a 3 phase short circuit. Thus

$$i_f(t) = -I_f(0) \left\{ \left[\frac{(L_d - L'_d)}{L'_d} \right] \left[\varepsilon^{-t/T'_d} - (1 - \frac{T_{kd}}{T''_d}) \varepsilon^{-t/T''_d} - \frac{T_{kd}}{T''_d} \varepsilon^{-t/T_a} \cos \omega t \right] \right\} \quad I_f(0) \qquad \text{(VI-379)}$$

A typical oscillogram of the field current under these conditions is shown in Figure VI-23.

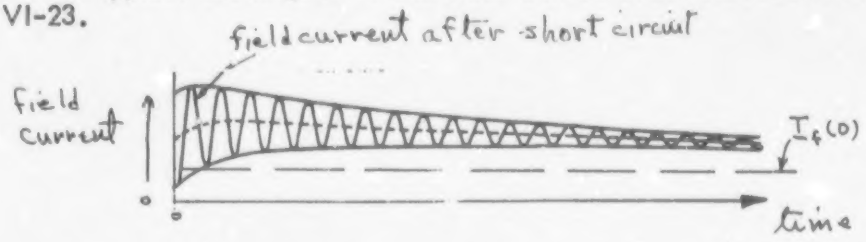


Figure VI-23. Typical Field Current of Alternator on Short Circuit

In the above derivation, it was assumed that the alternator was operating at a no load condition prior to t=0. If we wish to study the effects of initial load prior to the short circuit it is only necessary to determine $V_d(0)$ and $V_q(0)$ for that steady state condition and switch the negative of these values in the equations describing the behavior - the same procedure as used previously. The current solution, it will be recalled, is the change of current that results. This change of current is added to the steady state current existing at t=0 to obtain the total current as a function of time. Schematically, this situation is shown in Figure VI-24 by means of a "one line" diagram. "One line" diagrams are used to represent polyphase systems under balanced conditions. Note that this diagram represents an alternator connected to a bus through a transformer. The transformer is represented by its "through" impedance. At $t < 0$ the bus voltage is $V_{ta}(0)$ and the alternator is supplying a current $I_a(0)$ either directly to a load or into the remainder of a system which may be loads and other alternators.

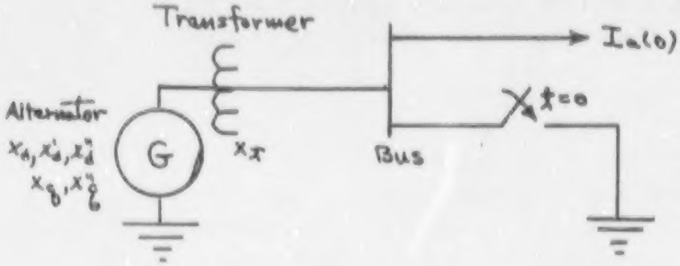

Transformer
$I_a(0)$
$t=0$
Alternator
x_d, x'_d, x''_d
x_q, x''_q
G
x_T
Bus

Figure VI-24. "One Line" Diagram Representation

At $t = 0$, the switch is closed simultaneously in all three phases and it is assumed, as before, that the speed remains constant during the period of time of the study.

The effect of the transformer impedance (negligible resistance) can be dealt with in the same fashion as in the steady state situations discussed in VI.8, i.e., it is added to each of the machine reactances because i_d and i_q both flow through this reactance. Thus,

$$X_d = x_d + x_t \qquad x_q = x_q + x_t$$
$$X'_d = x'_d + x_t \tag{VI-380}$$
$$X_d = x''_d + x_t \qquad X''_q = x''_q + x_t$$

The first step is to construct a steady state phasor diagram, based on $V_{ta}(0)$, I_a (0) and ϕ to determine $V_d(0)$, $V_q(0)$ as shown in Figure VI-25.

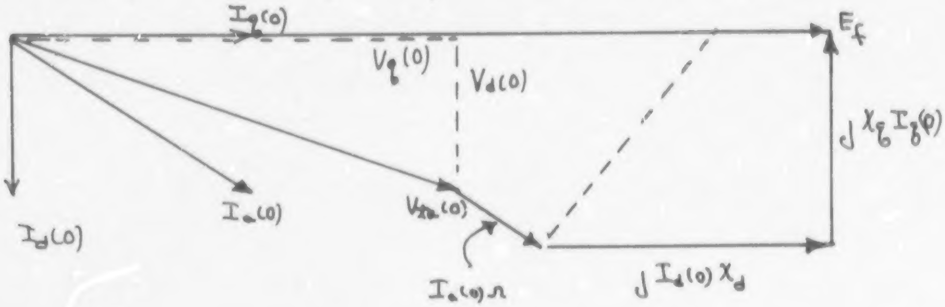

$I_q(0)$
E_f
$V_q(0)$
$V_d(0)$
$j X_q I_q(0)$
$I_a(0)$
$V_{ta}(0)$
$I_d(0)$
$I_a(0) \Omega$
$j I_d(0) X_d$

Figure VI-25. Steady State Phasor Diagram, $t < 0$.

The components of $V_{ta}(0)$ along the d and q axis are, of course, $V_d(0)$ and $V_q(0)$. These values are r.m.s. and must be converted to $v_d(0)$, $v_q(0)$ by multiplying $\sqrt{2}$ as in (VI-225). We can also find $i_d(0)$ and $i_q(0)$. After solving for $i_d(t)$, $i_q(t)$, as in the derivation for the no load case above, we add $i_d(0)$ and $i_q(0)$ and solve for $i_a(t)$.

VI.11 SWITCHED LOAD STUDIES: Problems involving transients which occur when loads are switched can be studied quite readily using the principles and equations from Section VI.10. For example, suppose it is desired to study the bus voltage variation when an impedance Z_{L2} is switched on an alternator supplying load current to an initial load represented by impedance, Z_{L1}. The one line diagram for this situation and its equivalent are shown in Figure VI-26.

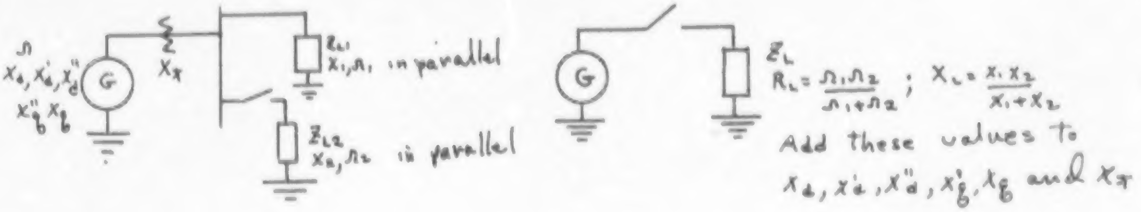


Figure VI-26. Equivalence of Situation

We can treat this as if it were an alternator on short circuit supplying an initial load (as in Section VI.10). After determining $V_d(0)$, $V_q(0)$, $i_d(0)$, and $i_q(0)$ the equations for the change in i_d and i_q can be obtained. Total i_a can then be calculated by adding to these the values of $i_d(0)$ and $i_q(0)$. It should be carefully noted and recognized that the equations for i_d and i_q were derived based on very low values of resistance which made several simplifying assumptions possible. If this is not the situation for the specific case being studied, one must go back and obtain new expressions for the change in current $i_d(t)$ and $i_q(t)$. However, most voltage transient studies of this type deal with the switching of loads, which appear very nearly completely reactive when switched on (for example, the starting of large motors, arc furnaces, etc.) After the value $i_a(t)$ is determined, the variation of $V_{ta}(t)$ can be obtained by taking the product of the series impedance equivalent (to the parallel combination Z_{1L} and Z_{2L}) and $i_a(t)$.

VI.12 SYNCHRONOUS MACHINE DYNAMICS: When shaft torque is changed - either by increasing load on a motor or changing the prime mover torque of an alternator, the rotor assumes a new position with respect to the stator field. That is, the power angle, δ, changes. It will go through a transient period before settling down to its new steady state position. The factors affecting this transient response are very complex as will be shown in this section. However, by making appropriate simplifying assumptions we can obtain results which will yield information of value about this period. Most machines exhibit under damping. That is, δ, oscillates about its steady state position with the magnitude of oscillations decaying exponentially. If shaft torque due to the connected device is also oscillating at frequencies near the "natural" frequency of the machine, very large amplitudes may result because of the resonance effect. In order to examine damping and frequency of oscillation we will assume the following:

1. The oscillations are small sinusoidal excursions about the steady state operating point (power angle)
2. Steady state speed, ω, is at rated speed
3. Excitation voltage and terminal voltage are constant
4. The effects of saturation are neglected
5. The applied stator phase "a" voltage is:

$$v_a = V_m \sin(\omega t + \alpha) \qquad \text{(VI-381)}$$

6. The stator rotating magnetic field position, θ_s, which will be a reference, is given by:

$$\theta_s = \omega t \qquad \text{(VI-382)}$$

In the above, α is the arbitrary angle necessary to relate stator field position and applied voltage. The angular position of the rotor, θ, is related to the stator field position by:

$$\theta = \theta_s - \delta = \omega t - \delta \qquad \text{(VI-383)}$$

This convention is for a motor, where the rotor axis lags the stator field axis. If the rotor is changing position, $\delta = \delta(t)$, and we can express instantaneous rotor angular velocity as:

$$\omega(t) = \frac{d\theta}{dt} = \omega - \frac{d\delta}{dt} \qquad \text{(VI-384)}$$

$\omega(t)$ denotes the actual angular velocity; ω is the steady state angular velocity. In terms of rotor angle, θ, we can express applied voltage (VI-381) as:

$$\begin{aligned} v_a &= V_m \sin(\theta + \delta + \alpha) \\ &= V_m \sin\theta \cos(\delta + \alpha) + V_m \cos\theta \sin(\delta + \alpha) \qquad \text{(VI-385)} \end{aligned}$$

From the transformation, (VI-384), v_a can be expressed in terms of v_d, v_q as:

$$v_a = v_d \cos\theta - v_q \sin\theta \qquad \text{(VI-386)}$$

From a comparison of (VI-385) and (VI-386) we see that:

$$v_d = V_m \sin(\delta + \alpha) \qquad \text{(VI-387)}$$

and:

$$v_q = - V_m \cos(\delta + \alpha) \qquad \text{(VI-388)}$$

From (VI-180) through (VI-190) the applicable equations for balanced conditions are, using (VI-387) and (VI-388):

$$v_d = V_m \sin(\delta + \alpha) = \frac{d\lambda_d}{dt} - \omega(t)\lambda_q + r\,i_d \qquad \text{(VI-389)}$$

$$v_q = - V_m \cos(\delta + \alpha) = \frac{d\lambda_q}{dt} + \omega(t)\lambda_d + r\,i_q \qquad \text{(VI-390)}$$

$$\frac{3}{2}(i_q\lambda_d - i_d\lambda_q) = T_{load} + J\frac{d\omega}{dt} \qquad \text{(VI-391)}$$

$$\lambda_d(s) = X_2(s)\, i_d(s) + X_3(s)\, v_f(s) \qquad \text{(VI-392)}$$

$$\lambda_q(s) = X_1(s)\, i_q(s) \qquad \text{(VI-393)}$$

where $X_1(s)$, $X_2(s)$ and $X_3(s)$ are as previously defined in (VI-186,-187,-188).

Now, assume an incremental load perturbation, ΔT, is introduced. The variables, i.e., currents, flux linkages, and load angle change by small increments, denoted by Δ , from the steady state values denoted by the subscript o. The total value of the variable is the sum of the incremental change and the steady state value. Thus,

$$\begin{aligned} i_q &= i_{qo} + \Delta i_q \\ i_d &= i_{do} + \Delta i_d \\ \lambda_d &= \lambda_{do} + \Delta\lambda_d \\ \lambda_q &= \lambda_{qo} + \Delta\lambda_q \\ \delta &= \delta_o + \Delta\delta \end{aligned} \qquad \text{(VI-394)}$$

where δ_o denotes steady state angle power angle minus the arbitrary angle, α , and $\omega(t)$ is as defined in (VI-384).

The first of the describing equations, (VI-389) becomes, using v_d from (VI-387) and noting that:

$$V_m \sin(\delta_o + \Delta\delta) = V_m \sin \delta_o \cos \Delta\delta + V_m \cos \delta_o \sin \Delta\delta \qquad \text{(VI-395)}$$

$$V_m \sin \delta_o \cos \Delta\delta + V_m \cos \delta_o \sin \Delta\delta = \frac{d}{dt}(\lambda_o + \Delta\lambda_d) +$$

$$- \left[\omega - \frac{d}{dt}(\delta_o + \Delta\delta)\right](\lambda_{qo} + \Delta\lambda_q) + r(i_{do} + \Delta i_d) \qquad \text{(VI-396)}$$

For small excursions of power angle:

$$\sin \Delta\delta \approx \Delta\delta \qquad \text{(VI-397)}$$

$$\cos \Delta\delta \approx 1.0 \qquad \text{(VI-398)}$$

Also, of course,

$$\frac{d\lambda_{do}}{dt} = 0; \ \frac{d\lambda_{qo}}{dt} = 0; \ \frac{d\delta_o}{dt} = 0 \qquad \text{(VI-399)}$$

Using these relationships and neglecting second order differentials, (VI-396) simplifies to:

$$+ V_m \sin \delta_o + V_m(\Delta\delta) \cos \delta_o = \frac{d}{dt}\Delta\lambda_d - \omega\lambda_{qo} - \omega\Delta\lambda_q + \lambda_{qo}\frac{d}{dt}\Delta\delta +$$

$$+ r\, i_{do} + r\Delta i_d \qquad \text{(VI-400)}$$

In the steady state, this relationship is:

$$+ V_m \sin \delta_o = - \omega\lambda_{qo} + r\, i_{do} \qquad \text{(VI-401)}$$

From (VI-185), with V_f constant

$$X_2(s)\,\Delta i_d(s) = \Delta\lambda_d(s) \qquad \text{(VI-402)}$$

Subtracting the Laplace Transform of (VI-401) from the Laplace Transform of (VI-400) and using (VI-402) for $\Delta i_d(s)$ yields:

$$(V_m \cos \delta_o - \lambda_{qo}s)\Delta\delta + \omega\Delta\lambda_q - \left(\frac{r}{X_2(s)} + s\right)\Delta\lambda_d = 0 \qquad \text{(VI-403)}$$

It should be noted that $X_3(s)$ is not in this equation because $v_f = 0$. Also the (s) designation for transformed variables has been dropped but is implied.

Similarly, for (VI-390), VI-391) for step torque change $\frac{\Delta T_L}{s}$, we have:

$$(V_m \sin \delta_o + \lambda_{do} s)\Delta\delta - \left(\frac{r}{X_1(s)} + s\right)\Delta\lambda_q - \omega\Delta\lambda_d = 0 \qquad \text{(VI-404)}$$

$$J\, s^2\, \Delta\delta + \frac{3}{2}\left[i_{qo} - \frac{\lambda_{qo}}{X_2(s)}\right]\Delta\lambda_d + \frac{3}{2}\left[\frac{\lambda_{do}}{X_1(s)} - i_{do}\right]\Delta\lambda_q = \frac{\Delta T_L}{s} \qquad \text{(VI-405)}$$

(VI-403, -404, -405), with appropriate initial conditions can be solved to yield a solution for $\Delta\delta(t)$. However, it should be noted that the expression will contain terms with s to the 5th power because of:

$$s\, X_1(s)\, X_2(s)\, s^2\, J \qquad \text{(VI-406)}$$

A computer is necessary to deal with the equations as written. In order to proceed, without a computer, some simplifying assumptions can be made. As an illustration, assume a step load of torque is impressed upon a synchronous motor under no load conditions for $t < 0$.

Thus:

$$\sin \delta_o = 0 \qquad i_{do} = 0 \qquad \text{(VI-407)}$$

$$\cos \delta_o = 1 \qquad i_{qo} = 0$$

$$\lambda_{qo} = 0$$

We will neglect the transformer type voltages, such as:

$$\frac{d\lambda_d}{dt}, \frac{d\lambda_q}{dt} \approx 0 \qquad \text{(VI-408)}$$

in this analysis. Also, with the assumptions used in (VI-276) and (VI-277)

$$\omega^2 + \frac{r^2}{X_1(s)X_2(s)} \approx \omega^2 \qquad \text{(VI-409)}$$

$$\frac{1}{X_1(s)} \approx \frac{1}{L''_q} \quad ; \quad \frac{1}{X_2(s)} \approx \frac{1}{L''_d} \qquad \text{(VI-410)}$$

From (VI-390), with $i_{qo} = 0$, (no load)

$$\lambda_{do} = \frac{v_q}{\omega} = -\frac{V_m}{\omega} \tag{VI-411}$$

With these approximations (VI-403, -404 and -405) become:

$$V_m \Delta\delta + \omega\Delta\lambda_q - \frac{r}{L''_d}\Delta\lambda_d = 0 \tag{VI-412}$$

$$\frac{V_m}{\omega} s\,\Delta\delta + \frac{r}{L''_q}\Delta\lambda_q + \omega\Delta\lambda_d = 0 \tag{VI-413}$$

$$J s^2 \Delta\delta - \frac{3 V_m}{2\omega L''_q}\Delta\lambda_q = \frac{\Delta T_L}{s} \tag{VI-414}$$

With these equarions:

$$\Delta\delta = \frac{\begin{vmatrix} 0 & \omega & -\frac{r}{L''_d} \\ 0 & \frac{r}{L''_q} & \omega \\ \frac{\Delta T_L}{s} & -\frac{3 V_m}{2\omega L''_q} & 0 \end{vmatrix}}{\begin{vmatrix} V_m & \omega & -\frac{r}{L''_d} \\ \frac{V_m}{\omega} s & \frac{r}{L''_q} & \omega \\ J s^2 & \frac{-3 V_m}{2\omega L''_q} & 0 \end{vmatrix}}$$

$$\Delta\delta = \frac{\left(\omega^2 + \frac{r^2}{L''_d\, L''_q}\right) \frac{\Delta T_L}{s}}{\omega^2\, J\, s^2 + \frac{r^2}{L''_q L''_d}\, J\, s^2 + \frac{3\, r\, V_m^2}{2\, \omega^2 L''_d\, L''_q} + \frac{3\, V_m^2}{2\, L''_q}} \qquad \text{(VI-415)}$$

$$\approx \frac{2\, X''_q \omega\, \Delta T_L}{3\, V_m^2\, s} \left\{ \frac{1}{\frac{2\, \omega\, J\, X''_q}{3\, V_m^2}\, s^2 + \frac{r}{\omega X''_d}\, s + 1} \right\} \qquad \text{(VI-416)}$$

This is of the form:

$$\frac{\Delta T_L}{\nu^2\, J\, s} \left\{ \frac{1}{\frac{s^2}{\nu^2} + \frac{2\xi}{\nu}\, s + 1} \right\} \qquad \text{(VI-417)}$$

where

ν = natural undamped angular velocity of oscillation

$$= V_m \sqrt{\frac{3}{2\, \omega\, J\, X''_q}} \qquad \text{(VI-418)}$$

ξ = the damping ratio

$$= \frac{r}{2}\, \frac{V_m}{\omega X''_d} \sqrt{\frac{3}{2\, \omega\, J\, X''_q}} \qquad \text{(VI-419)}$$

The inverse transform of (VI-417) yields:

$$\Delta\delta(t) = \frac{\Delta T_L}{\nu^2\, J} \left\{ 1 + \frac{1}{\sqrt{1 - \xi^2}} \cdot \varepsilon^{-\xi \nu t} \sin\left(\nu \sqrt{1 - \xi^2}\, t - \psi\right) \right.$$

$$\text{where} \quad \psi = \tan^{-1} \frac{\sqrt{1 - \xi^2}}{-\xi} \qquad \text{(VI-420)}$$

From (VI-420) several observations can be made:

1. The relationship between steady state change in power angle and change in torque is:

$$C_s \ \Delta\delta = \Delta T_L \text{ where } C_s = \nu^2 J \qquad \text{(VI-421)}$$

2. The damping effect approaches zero as $r \to 0$.

3. The maximum amplitude of the oscillation is given by (approximately)

$$\frac{\Delta T_L}{\nu^2 J\sqrt{1-\xi^2}} \approx \frac{\Delta T_L}{\nu^2 J} \qquad \text{(VI-422)}$$

4. The time constant for the decay of sinusoidal oscillations is:

$$T = \frac{1}{\xi\nu} = \frac{4\,\omega^2 X_d'' X_q'' J}{r\,V_m^2} \qquad \text{(VI-423)}$$

5. Now $V_m \propto V_f$. If either r or V_f are small, the time constant for damping out the oscillations approaches infinity and the machine tends to "hunt". This is indeed the situation for an under excited synchronous motor.

If $\Delta T(t)$ was a sinusoidal function, rather than a step function, $\Delta\delta(t)$ would oscillate at two angular velocities - one corresponding to that of the driving torque and the other the natural angular velocity ν found here. If these two angular velocities have the same value, resonance results and very large amplitudes of oscillation will be experienced.

It should be noted that the analysis and representation above is for small oscillations and for the machine at no load. Results are not accurate for other conditions, however; the analysis does indicate which parameters influence the synchronous machine dynamics.

Note that we can "work backwards" from (VI-417) and find the differential equation corresponding to the transformed equation (VI-417). Thus:

$$\Delta\delta = \frac{\Delta T_L/s}{J\,s^2 + 2J\xi\nu_s + \nu^2 J} \qquad \text{(VI-424)}$$

From (VI-421):

$$\nu^2 J = C_s \qquad \text{(VI-421)}$$

and we will define:

$$2\,J\,\xi\nu = C_d \tag{VI-425}$$

and we can write (VI-424) as a function of time as:

$$J\,\frac{d^2\Delta\delta}{dt^2} + C_d\,\frac{d\Delta\delta}{dt} + C_s\,\Delta\delta = \Delta\,T_L(t) \tag{VI-426}$$

(VI-426) is widely used as the describing equation for the machine. In (VI-426):

$J\,\frac{d^2\Delta\delta}{dt^2}$ is the inertia torque

$C_s\,\Delta\delta$ is the motor torque produced by synchronous motor action

$C_d\,\frac{d\Delta\delta}{dt}$ is a damping torque and acts to resist change in $\Delta\delta$. Actually, it is an asynchronous torque identical to the torque in an induction motor. It results from an interaction between currents in the damper windings (KD and KQ) on the rotor and the stator field.

VI.13 SYNCHRONOUS MACHINE PARAMETERS BY TEST: If we refer to Figure VI-21 we see that at least two of the three phases (and possibly all three phases) have a unidirectional component of current in the short circuit current obtained by test. If we remove this unidirectional component, we have the fundamental component shown as phase "a" in the oscillograms of Figure VI-21. The envelope is plotted in Figure VI-22. Comparison with equation (VI-344) indicates that:

$$oa = \frac{\sqrt{2}\,V_{ta}}{X''_d}$$

$$ob = \frac{\sqrt{2}\,V_{ta}}{X'_d} \tag{VI-427}$$

$$oc = \frac{\sqrt{2}\,V_{ta}}{X_d}$$

and that the subtransient portion decays with time constant T''_d, whereas the transient portion decays with time constant T'_d. Similarly, from a recording of $i_f(t)$, we can determine T_a.

The Institute of Electrical and Electronic Engineers publication "Test Procedures for Synchronous Machines", # 115 details procedures for obtaining the above parameters as well as other parameters used in machine analysis. Applicable sections for the above and othes are:

Section 7.15 - for X_d
7.20 - for X_q
7.25 - for X'_d
7.30 - for X''_d
7.35 - for X''_q
7.65 - for T'_{do}
7.70 - for T'_d
7.75 - for T''_{do}
7.80 - for T''_d

Note that T_a can also be calculated from:

$$T_a = \left(\frac{2}{r}\right) \frac{L''_d \; L''_q}{L''_q + L''_d} \qquad \text{(VI-428)}$$

The test code does not specify how to test for T_{KD}, T''_q, and T''_{qo}, which are necessary in some types of analysis. The quantities must be calculated or obtained from the machine designer. Two classic references on the prameters are: "Calculation of Synchronous Machine Constants", by L. A. Kilgore, Trans. AIEE, Vol. 50, 1931, pg. 1201; "Determination of Synchronous Machine Constants by Test", S. H. Wright, Trans. AIEE, Vol. 50, 1931, pg. 1331.

VI.14 SYNCHRONOUS MOTOR APPLICATION: Synchronous motors, in general, are more expensive than induction, or asynchronous, motors except in the case of motors designed to run at relatively slow speeds. For example, consider an application involving a load to be driven at around 150 rpm. For a 60 cps source, the motor must have 48 poles. Mechanically, a 48 pole induction motor is difficult to design, whereas a 48 pole synchronous machine is possible without complication. Thus, the choice involves a higher speed induction motor with speed reducing provision (either belt and pulleys or gearing) or a synchronous motor with associated d.c. supply and control for the field circuit. The synchronous motor starts with field shorted and excitation is applied as it nears synchronous speed. Control circuitry is necessary to insure the application of field voltage, or excitation at the proper time.

The decision as to whether to apply a synchronous or an induction motor to a specific drive requirement is usually one of economics. Seldom does a requirement for absolutely constant speed enter into the decision in power applications. A general rule of thumb is that "when the horsepower exceeds the rpm, the synchronous motor and control are economically advantageous". It should be emphasized that this is completely general. The power factor correction capability of the synchronous motor can be decisive in an economic evaluation of which motor to buy. In a situation where one desires to obtain a higher plant power factor while adding motor capability, the higher cost of the synchronous motor, d.c. excitation provision, and control must be evaluated against the cost of the induction motor, its control and static capacitors which yield the same power factor correction capability as the synchronous motor. The following example will illustrate the procedure for this type of evaluation:

Example

Assume an industrial plant with an existing connected load of 10000 Kw at a power factor of 0.8 lagging. It is necessary to add 3000 Kw of motor load (one unit) and it is desired to improve the overall plant power factor to 0.9. Three possibilities are open. They are:

1. An induction motor - with static capacitors

2. A synchronous motor - possibly with static capacitors
   a. a unity p.f. capable motor
   b. an 0.8 p.f. (leading) capable motor

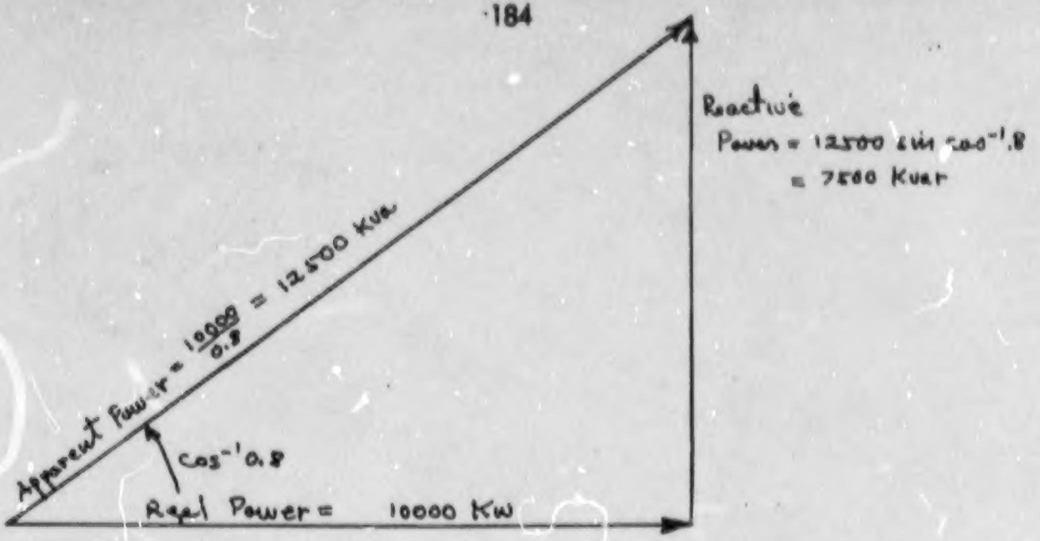
Assume that either motor is 93% efficient and that the induction motor takes power at 0.85 power factor. A volt-ampere, power diagram for the existing situation in the plant is shown in Figure VI-27a. Figure VI-27b depicts the induction motor. The combination of existing plant plus induction motor is shown in Figure VI-27c. Also, the power-reactive power values for a 0.9 power factor after adding the 3000 hp motor are shown. It can be seen that for the new plant load of 13,230 Kw the Kvar permissible is equal to 6420 Kvar. However, with the original Kvar plus that added by the induction motor the new total without static capacitor correction would be 7500 + 2020 = 9520 Kvar. Since 6420 Kvar yields 0.9 p.f., 9520 - 6420 = 3100 Kvar of static capacitors will be required. If a unity power factor capability synchronous motor were added, rather than the induction motor, only 7500-6420 = 1080 Kvar of static capacitors would be required.

Figure VI-28 indicates the overall effect of using an 0.8 p.f., leading capability motor.

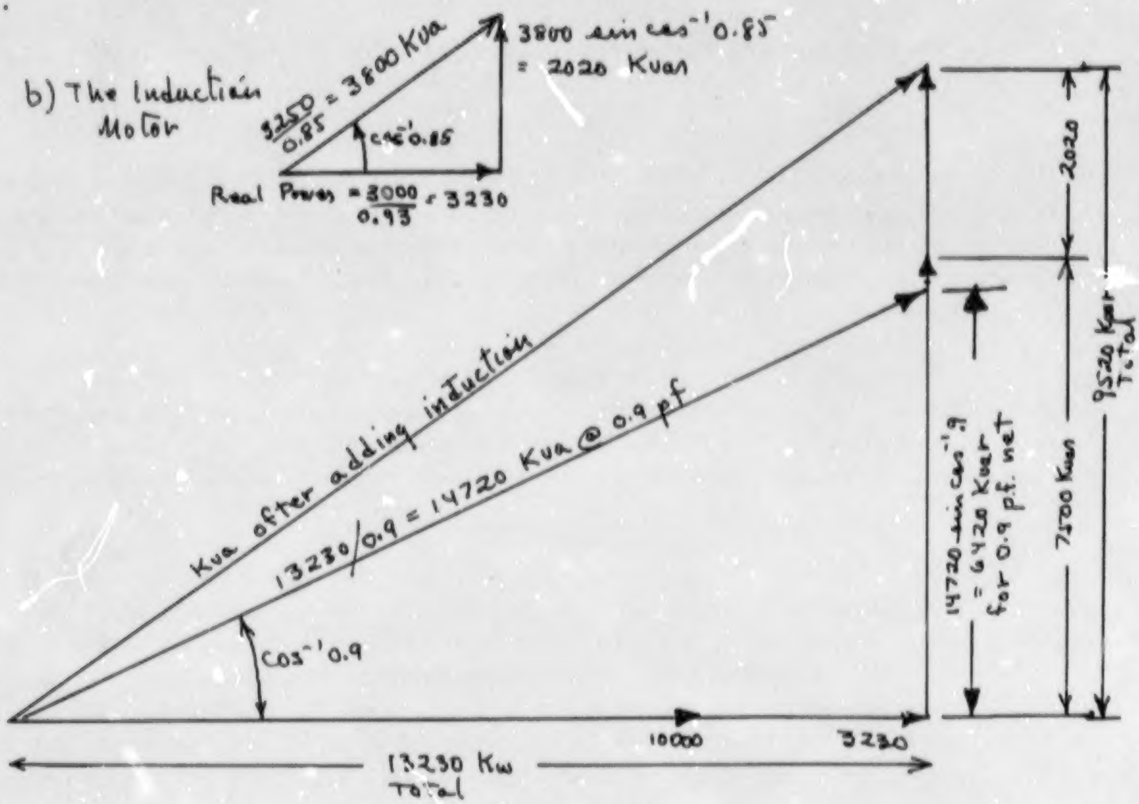
The synchronous motor would act as a capacitor and supply 2420 Kvar of leading power factor reactive power resulting in an overall plant power factor - after adding the motor - of:

$$\text{p.f.} = \cos \tan^{-1} \frac{5080}{13230} = 0.945$$

Thus, the power factor has been improved to better than 0.9 without static capacitors. Exactly which choice is to be specified would depend upon relative cost of the motors and capacitors.


Reactive Power = 12500 sin cos⁻¹.8 = 7500 Kvar
Apparent Power = 10000/0.8 = 12500 Kva
cos⁻¹0.8
Real Power = 10000 KW

a) Existing Plant


3250/0.85 = 3800 Kva
3800 sin cos⁻¹0.85 = 2020 Kvar
cos⁻¹0.85
Real Power = 3000/0.93 = 3230
Kva after adding induction
13230/0.9 = 14720 Kva @ 0.9 pf
cos⁻¹0.9
2020
14720 sin cos⁻¹.9 = 6420 Kvar for 0.9 p.f. net
7500 Kvar
9520 Kvar Total
10000
3230
13230 Kw Total

b) The Induction Motor

c) The total Plant

Figure VI-27. The Power-Reactive Power Situation Using an Induction Motor.

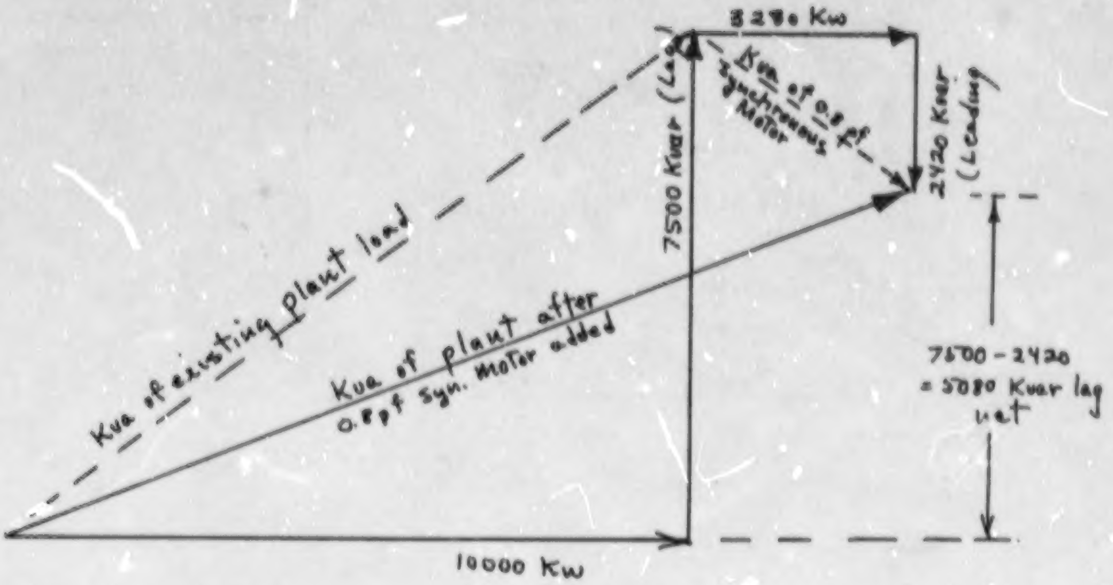

3230 Kw
7500 Kvar (Lag)
Kva of 0.8 pf synchronous Motor
2420 Kvar (Leading)
Kva of existing plant load
Kva of plant after 0.8 pf Syn. motor added
7500 - 2420
= 5080 Kvar lag net
10000 Kw

Figure VI-28. Power-Reactive Power Situation Using an 0.8 p.f. Synchronous Motor

VI.15 SYNCHRONOUS MACHINE SIMULATION AND MODELING: If one desires to carry the block diagram modeling of a synchronous machine to the point of interconnection of the various block diagrams formulated thus far (see Figures VI-7,-8, and -9) one must specify whether it is a motor or alternator that is to be modeled, and the system configuration.

For example, a single alternator can serve a combination of loads, it can be paralleled with other alternators of more or less equal (total) capacity, or it can operate in parallel with an 'infinite bus' as previously defined. Each situation is a particular case. We will examine the simplest case in order to illustrate the method and procedure.

Consider a single alternator, driven at constant rated speed, supplying a balanced three phase load consisting of $R_L + j X_L = Z_L$ ohms (equivalent) in each phase. Assume that the object of our investigation will be the variation of terminal voltage when load is switched on and that a voltage regulator is used to assist in the regulation of terminal voltage. We can assume a block diagram for the voltage regulating circuit and the d.c. excitation system as shown in Figure VI-29. Recall that for a balanced system, the zero sequence voltage and currents are zero and that, from (VI-215),

$$|v_a|^2 = \sqrt{v_d^{\,2} + v_q^{\,2}} \qquad \text{(VI-429)}$$

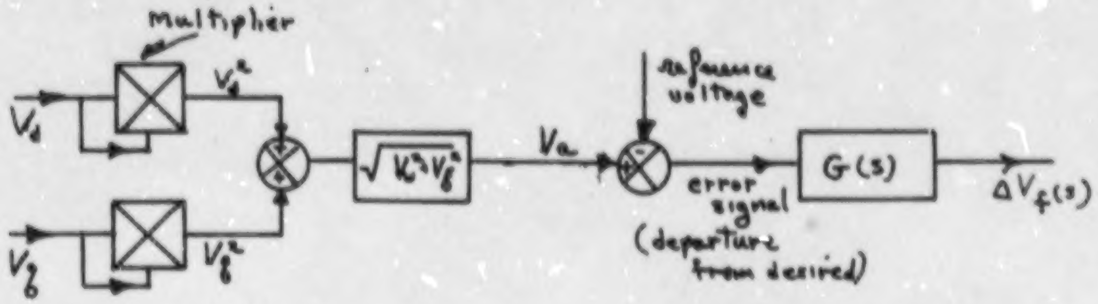

multiplier
V_d
V_d^2
V_q
V_q^2
$\sqrt{V_d^2+V_q^2}$
V_a
reference voltage
error signal
(departure from desired)
G(s)
$\Delta V_f(s)$

Figure VI-29. Voltage-Regulator-Excitation System

G(s) is the transfer function relating error signal input and output, Δv_f. If this were a regulating system utilizing an amplidyne, for example, the actual function for G(s) could be determined by modifying (V-25) by multiplying $I_a(s)$ by the transformed impedance of the alternator field winding.

Note that inputs into the block diagrams are, in addition to $\Delta v_f(s)$, $i_d(s)$ and $i_q(s)$. We can obtain these as follows: (where R_L, L_L is the equivalent series resistance and inductance of the load)

$$v_a = R_L\, i_a + L_L \frac{d\, i_a}{dt} \qquad \text{(VI-430)}$$

or, using the a, b, c - d, q, o transformation

$$v_d \cos\theta - v_q \sin\theta = (R_L + L_L \frac{d}{dt})(i_d \cos\theta - i_q \sin\theta) \qquad \text{(VI-431)}$$

which yields, when expanded:

$$v_d \cos\theta - v_q \sin\theta = R_L i_d \cos\theta - \omega L_L\, i_q \cos\theta + L_L \frac{d\, i_d}{dt} \cos\theta +$$

$$+ R_L i_q \sin\theta - \omega L_L i_d \sin\theta - L_L \frac{d\, i_q}{dt} \sin\theta \qquad \text{(VI-432)}$$

Equating coefficients of like trigonometric terms and taking the Laplace transform, considering the initial conditions as zero, yields the following relationship for the change in variables: $(\omega L_L = X_L)$

$$(R_L + L_L s)\ \Delta i_d(s) - X_L\ \Delta i_q(s) = \Delta v_d(s) \qquad \text{(VI-433)}$$

$$+ X_L\ \Delta i_d(s) + (R_L + L_L s)\ \Delta i_q(s) = + \Delta\ v_q(s) \qquad \text{(VI-434)}$$

Solving for $\Delta\ i_d(s)$, $\Delta i_q(s)$ yields:

$$\Delta\ i_d(s) = \frac{(R_L + L_L s)\Delta v_d(s) + X_L\ \Delta\ v_q(s)}{L_L s^2 + 2R_L L_L\ s + R_L{}^2 + X_L{}^2} \qquad \text{(VI-435)}$$

$$i_q(s) = \frac{-\ X_L\ \Delta\ v_d(s) + (R_L + L_L s)\ \Delta\ v_q(s)}{L_L s^2 + 2R_L L_L s + R_L{}^2 + X_L{}^2} \qquad \text{(VI-436)}$$

In order to obtain $i_q(0)$, $i_d(0)$, we can consider the steady state situation (Load before switching; $R + j\ X = Z$)

$$\frac{v_d(0) + j\ \ v_q(0)}{R + j\ X} = i_d(0) + j\ i_q(0) \qquad \text{(VI-437)}$$

$$i_d(0) + j\ i_q(0) = \frac{R\ v_d(0) + X\ v_q(0) + j[R\ v_q(0) - X\ v_d(0)]}{Z^2} \qquad \text{(VI-438)}$$

from which:

$$i_d(0) = \frac{R}{Z^2}\ v_d(0) + \frac{X}{Z^2}\ v_q(0);\ i_q(0) = \frac{R}{Z^2}\ v_q(0) - \frac{X}{Z^2}\ v_d(0) \qquad \text{(VI-439)}$$

The block diagram for (VI-432,-438) is shown in Figure VI-30:

We must obtain block diagrams relating $v_d(s)$ $v_q(s)$ to $\lambda_d(s)$, $\lambda_q(s)$, $i_d(s)$ and $i_q(s)$. From before, except changing the sign of current to conform to generator convention:

$$\Delta\ v_d(s) = s\,\Delta\lambda_d(s) - \Delta\lambda_q(s) - r\,\Delta\, i_d(s) \qquad \text{(VI-440)}$$

$$\Delta\, v_q(s) = s\Delta\lambda_q(s) + \Delta\lambda_d(s) - r\Delta\ i_q(s) \qquad \text{(VI-441)}$$

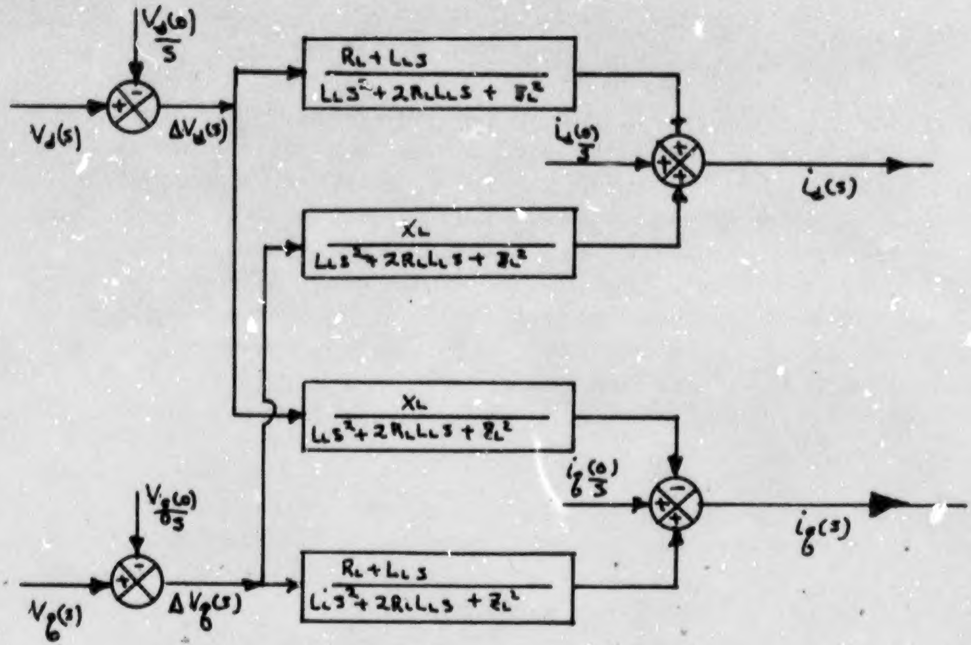

V_d(0)/s
V_d(s)
ΔV_d(s)
R_L + L_L s / L_L s^2 + 2R_L L_L s + Z_L^2
X_L / L_L s^2 + 2R_L L_L s + Z_L^2
i_d(0)/s
i_d(s)
X_L / L_L s^2 + 2R_L L_L s + Z_L^2
i_q(0)/s
i_q(s)
V_q(0)/s
V_q(s)
ΔV_q(s)
R_L + L_L s / L_L s^2 + 2R_L L_L s + Z_L^2

Figure VI-30. Voltage to Current Relationship, Balanced Load of $R_L + j\,\omega L_L = Z_L$ per phase

The block diagram for (VI-440 and -441) is shown in Figure VI-31.

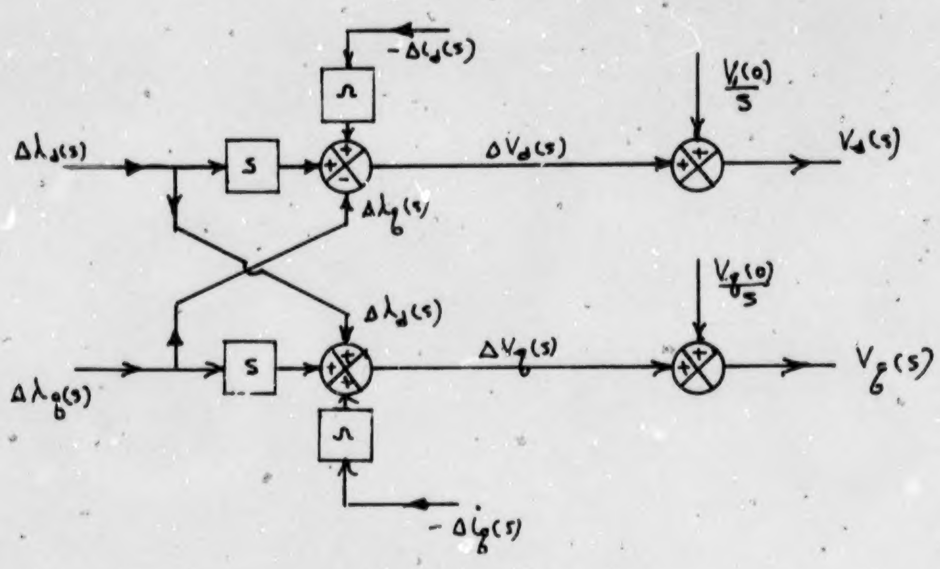

-Δi_d(s)
Ω
Δλ_d(s)
s
ΔV_d(s)
V_d(0)/s
V_d(s)
Δλ_q(s)
Δλ_d(s)
V_q(0)/s
ΔV_q(s)
V_q(s)
Δλ_q(s)
s
Ω
-Δi_q(s)

Figure VI-31. Voltage, Current and Flux Linkage Relationship.

Figure VI-8, -9, -29, -30 and -31 are interconnected in Figure VI-32 to form the complete system of alternator, load, voltage regulator and excitation source for the system assumed. (Note that the sign of i_d, i_q is reversed to conform to alternator convention). The only steady state value at $t < 0$ required here is that of $v_a(0)$. If this system is to be simulated on an analog computer the necessity to differentiate $\Delta\lambda_d$ and $\Delta\lambda_q$ will be a complication because exact differentiation using analog elements is not possible because of their finite output voltage limits. The same complication is present to a lesser extent in the load representation. It is necessary to either resort to "approximate" differentiation or to neglect the d/dt terms. The reader can refer to a text dealing with analog computer techniques for a dicsussion of the "apprxomate differentiator".

To illustrate modeling and simulation of a synchronous motor, we will choose a system involving a variable frequency, variable voltage to be impressed upon the synchronous motor in order to obtain a variable speed drive system. Such a system in its simplest form, would involve a rectifier-inverter combination energized from a constant potential, constant frequency source. Referring to (II-25), we observe that we must maintain a constant ratio of volts impressed per cycle in order to maintain a constant value of flux (and flux density) in the magnetic circuit. Since the machine is designed for some maximum value, it is essential that we maintain this ratio constant below its maximum value. A power supply for accomplishing this is shown in block diagram form in Figure VI-33.

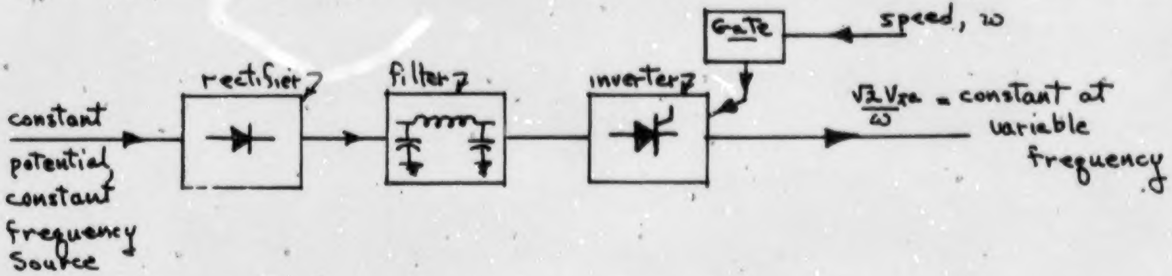

Gate
speed, ω
rectifier
filter
inverter
constant potential constant frequency Source
$\frac{\sqrt{2}V_{ta}}{\omega}$ = constant at variable frequency

Figure VI-33. Frequency Converter Block Diagram

In the converter, the input is rectified and filtered to provide direct current. The inverter consists of thyristors and diodes. The thyristors are switched in a definite sequence by the "gate" circuitry to form a 3 phase output. The gate switches at a frequency proportional to the speed signal (since speed and frequency in synchronous machines are proportional) and at such an instant within the cycle so as to provide a magnitude proportional to speed. In this course, we will not go into the details of the inverter and gate circuitry other than to note that we can "process" the d.c. to form a three phase output wherein we obtain a magnitude such that $\sqrt{2}\, V_{ta}/\omega$ = constant, where ω = actual speed.

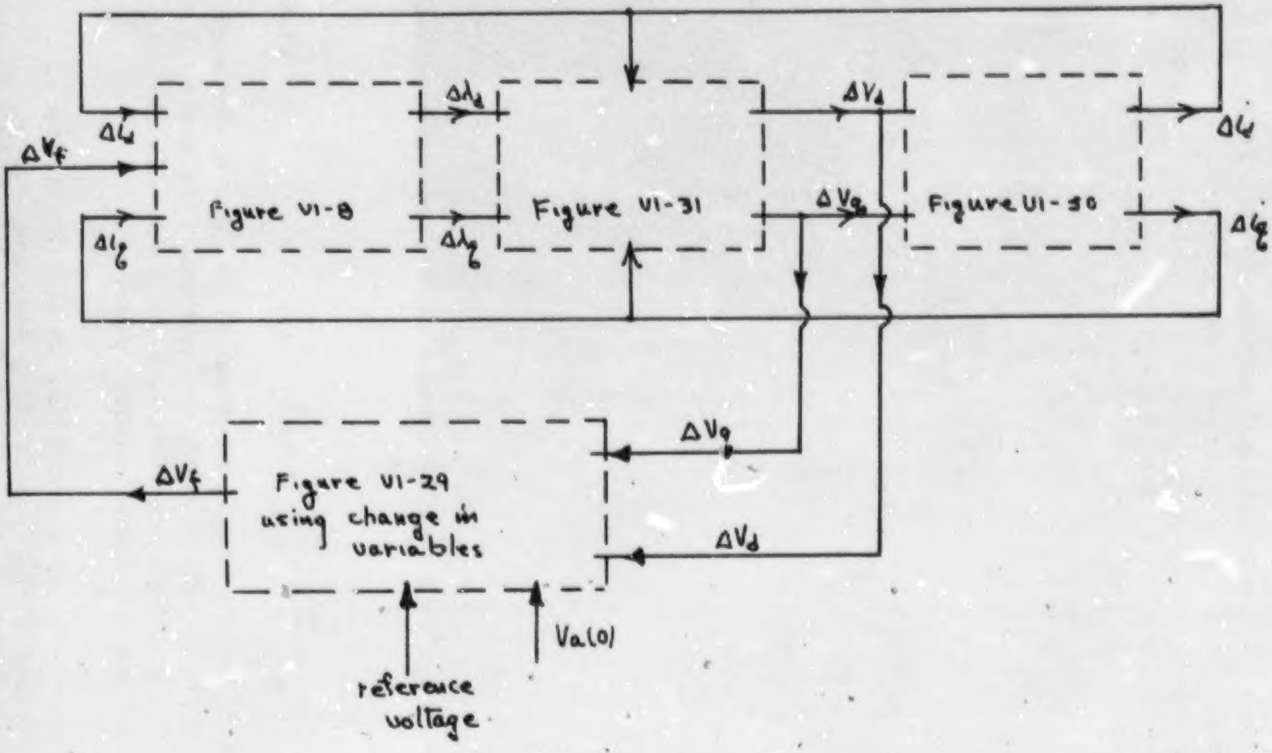

Δi_d
ΔV_f
Δi_q
Figure VI-8
Δλ_d
Δλ_q
Figure VI-31
ΔV_d
ΔV_q
Figure VI-30
Δi_d
Δi_q
ΔV_q
ΔV_f
Figure VI-29 using change in variables
ΔV_d
reference voltage
Va(0)

Figure VI-32. Interconnection for System Simulation

We will now rearrange the applicable equations to obtain block diagrams which we can then interconnect to form a model of the synchronous motor and load. We will neglect the $d\lambda/dt$ terms in the equations in order to simplify the simulation. From (VI-108) the applicable equations are (neglecting the $d\lambda/dt$ terms and recognizing T_L, V_f and $\sqrt{2}\, V_{ta}/\omega = v_a/\omega$ as inputs);

$$v_d + \omega\lambda_q = r\, i_d \tag{VI-442}$$

$$v_q = - \omega\lambda_q = r\, i_q \tag{VI-443}$$

$$\Delta\lambda_q = \Delta i_q\, X_1(s) \tag{VI-444}$$

$$\Delta\lambda_d = \Delta i_d\, X_2(s) + \Delta V_f\, X_3(s) \tag{VI-445}$$

$$T_e - T_L = \frac{3}{2}(\lambda_d\, i_q - \lambda_q\, i_d) - T_L = J\,\frac{d\omega}{dt} \tag{VI-446}$$

In addition, we will use the relationships:

$$(\frac{v_a}{\omega})^2 = (\frac{\sqrt{2}\, V_{ta}}{\omega})^2 = (\frac{v_d}{\omega})^2 + (\frac{v_q}{\omega})^2 \tag{VI-447}$$

and

$$T_e = \frac{P_e}{\omega} = v_q\, i_q + v_d\, i_d \tag{VI-448}$$

In the last equation P_e is instantaneous electrical power input to the motor and is the sum of the products of the in phase currents and voltages. The ohmic loss is neglected (as in the change in stored field energy because the $d\lambda/dt$ terms are not included above). Friction and windage losses are included as part of T_L.

Assuming we have currents i_d and i_q available, we can obtain speed, ω, for any load torque as shown in Figure VI-34) from (VI-444,-445, and -446):

From Figure VI-34, if we have currents i_d, i_q and initial values of the currents, and excitation available we can obtain ω, λ_q, and λ_d as outputs for given v_f, T_L.

Next, assume we have v_d and v_q. We can use the ω, λ_q and λ_d values from (VI-34) to obtain i_d and i_q using (VI-442, -443). This is shown in Figure VI-35.

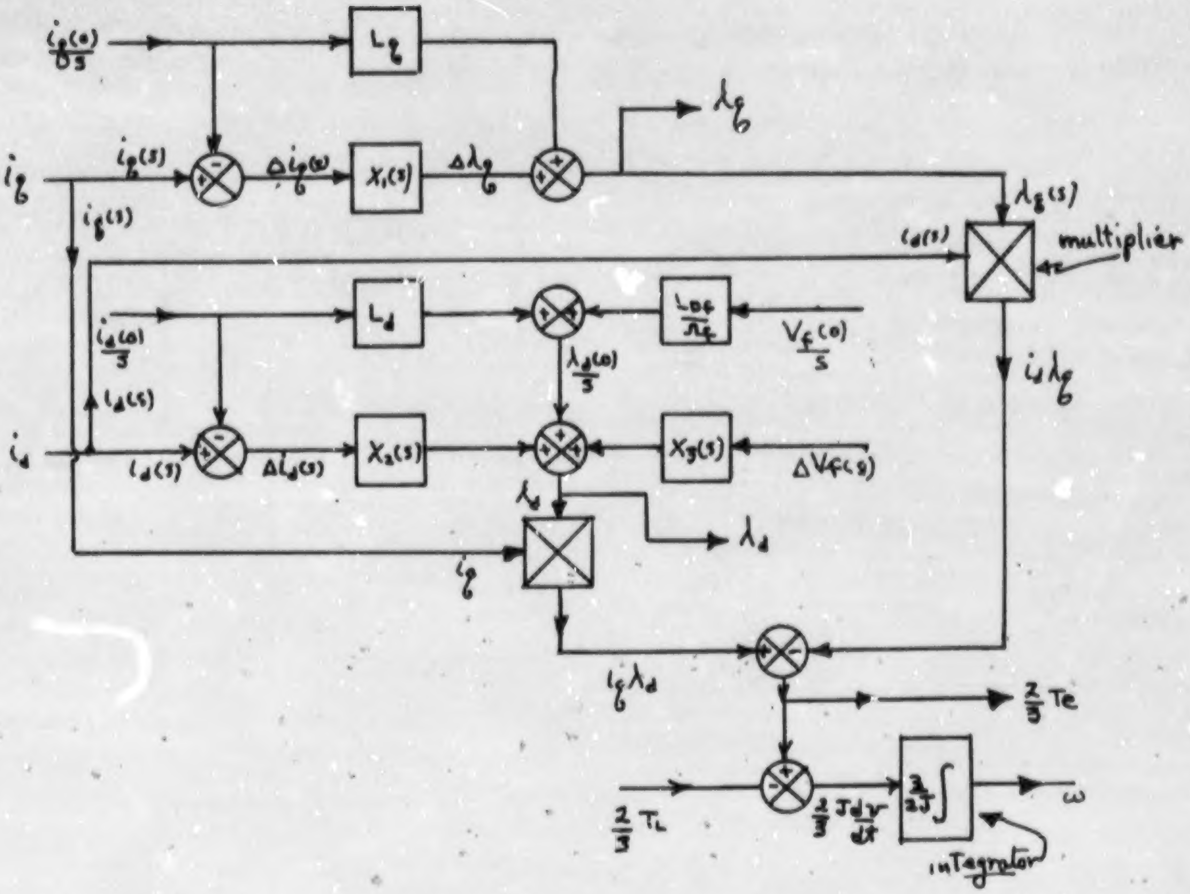

iq(0)/s
Lq
λq
iq
iq(s)
Δiq(s)
X1(s)
Δλq
λq(s)
id(s)
multiplier
iq(s)
Ld
LDf/rf
Vf(0)/s
id(0)/s
λd(0)/s
id(s)
idλq
id
id(s)
Δid(s)
X2(s)
X3(s)
ΔVf(s)
λd
λd
iq
iqλd
2/3 Te
2/3 TL
2/3 J dω/dt
3/2J ∫
ω
integrator

Figure VI-34. Speed, Torque, Current Relationships

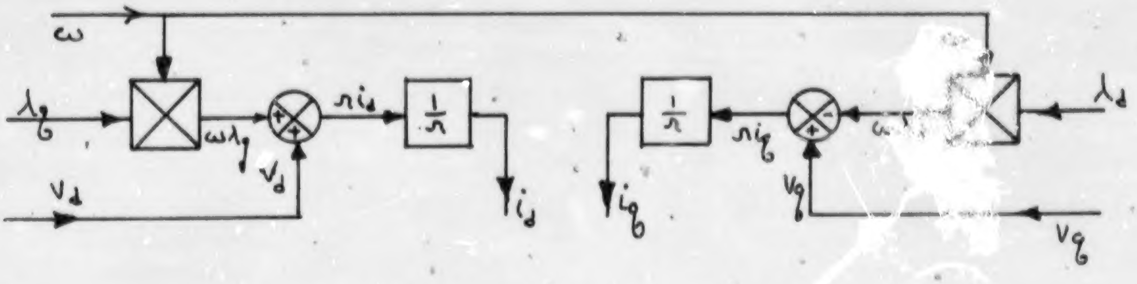

ω
λq
ωλq
vd
vd
rid
1/r
id
1/r
riq
iq
vq
vq
ωλd
λd

Figure VI-35. To obtain i_d, i_q.

We can now use the remainder of the equations above to find v_d, v_q. This is shown in Figure VI-36.

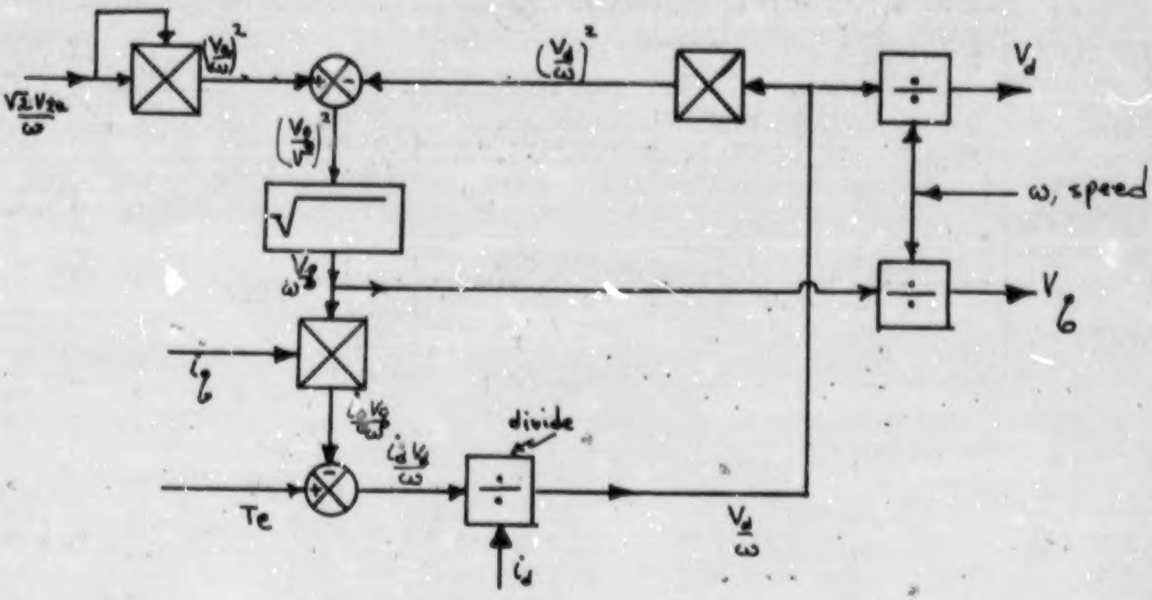
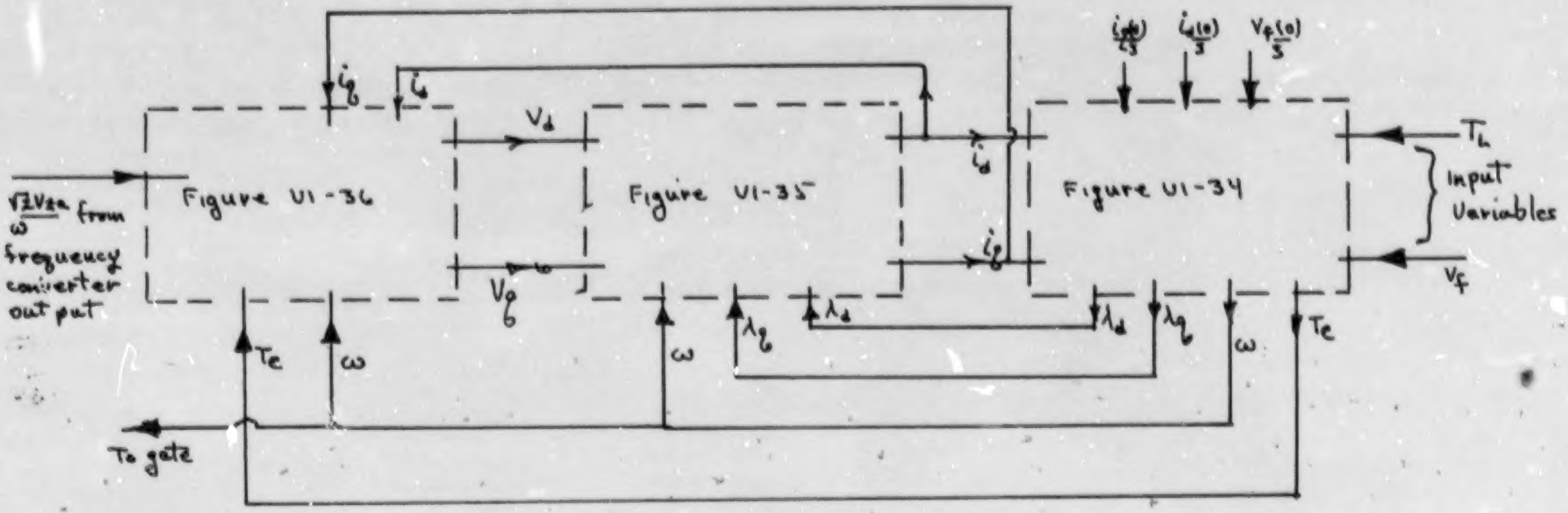

$\frac{\sqrt{2}V_{2a}}{\omega}$
$\left(\frac{V_a}{\omega}\right)^2$
+
-
$\left(\frac{V_d}{\omega}\right)^2$
V_d
$\left(\frac{V_q}{\omega}\right)^2$
$\sqrt{}$
ω, speed
$\frac{V_q}{\omega}$
V_q
i_q
$\frac{i_q V_q}{\omega}$
divide
$\frac{i_d V_d}{\omega}$
-
+
Te
÷
i_d
$\frac{V_d}{\omega}$

Figure VI-36. To obtain V_d, V_q

Interconnection of Figures (VI-34,-35 and -36) to obtain the complete model of the synchronous motor, with the assumptions given, is shown in Figure VI-37.



$\frac{\sqrt{2}V_{2a}}{\omega}$ from frequency converter output
Figure VI-36
Figure VI-35
Figure VI-34
i_q
i_d
V_d
V_q
i_d
i_q
$\frac{i_q(0)}{3}$
$\frac{i_d(0)}{3}$
$\frac{V_f(0)}{3}$
T_L
Input Variables
V_f
T_e
ω
ω
λ_q
λ_d
λ_d
λ_q
ω
T_e
To gate

Figure VI-37. Interconnection for Synchronous Motor Model

CHAPTER VII - POLYPHASE ASYNCHRONOUS MACHINES

VII.1 INTRODUCTION AND CONSTRUCTION FEATURES: Asynchronous machines are alternating current machines that do not run at synchronous speed. The most common form of this type machine is the "induction" machine. Although the machine can function as either a motor or generator (that is, the energy conversion process is reversible) the most widely used configuration is that of motor. Reasons for this will be discussed later.

The armature (usually the stator) of the induction machine and the synchronous machine are identical. However, the other member (usually the rotor) is quite different. The synchronous machine has a coil, energized from a direct current source, which results in a rotor field stationary with respect to the rotor. Since the rotor and stator fields must be stationary with respect to each other in order to produce average torque, the rotor must revolve at synchronous speed. The induction machine has a polyphase winding, rather than a coil, on the rotor. The excitation results from current flowing in the rotor winding. Current flows as a result of polyphase voltages existing either by induction and/or by injection. By far the most common variety is one in which induction accounts for the voltage and resulting current. The analysis in this text will deal with this configuration. The polyphase rotor may be either "wound" or "squirrel cage". The "wound" rotor has a regular polyphase winding similar to that on the stator, i.e., it must have the same number of phases and poles. The terminals of the rotor winding are connected to insulated slip rings mounted on the shaft. Carbon brushes hearing on the slip rings make the terminals available externally (and stationary) in order to add additional (external) resistance to the winding. This capability permits the user to alter the speed torque characteristic, as will be shown later. (If voltages of a special frequency are to be injected, they can be injected through these brushes). The squirrel cage rotor is so named because it resembles, in some respects, a cylindrical squirrel cage. This winding consists of conducting bars imbedded in slots in the rotor iron and short circuited at each end by conducting end rings. This construction is simple, relatively inexpensive and very rugged. It results in a widely used motor.

It can be shown that the induced voltages and resulting currents in a squirrel cage rotor are such as to result in a rotating mmf similar to that obtained from a wound type polyphase winding.

In order to visualize the torque-speed behavior, consider that the rotor is stationary. By electromagnetic induction, a voltage (depending upon the turns ratio between rotor and stator winding) is induced in the rotor. This action results from the stator field rotating past the rotor inductors and inductng a speed type voltage in the rotor. Since the winding is closed upon itself, currents flow in the rotor and a revolving field is established on the rotor (because it is a polyphase winding). Since the rotor and stator have the same number of phases and poles and the frequency in each is the same (with the rotor stationary) the fields are stationary with respect to each other and torque results. Since the rotor is free to turn, it accelerates.

Now, consider the situation at some speed. Let:

ω = mechanical angular velocity of the rotor

ω_s = the angular velocity of the stator field

ω_r = the angular velocity of the rotor field with respect to the rotor.

The relative speed between stator field and the rotor structure can be expressed as a fraction of stator field speed and is defined as the "slip". Thus:

$$\sigma = \frac{\omega_s - \omega}{\omega_s} \qquad \text{(VII-1)}$$

where:

σ = slip

Since at any speed, ω, the relative speed between rotor inductors and ω_s is only σ times as great as at standstill, it follows that the frequency of the voltage and currents in the rotor will only be the fraction σ of their value at standstill. Thus the revolving field of the rotor, ω_r, with respect to the rotor will be:

$$\omega_r = \sigma \omega_s \qquad \text{(VII-2)}$$

If we add ω_r, the speed of the field with respect to the rotor, to the rotor mechanical speed, ω, we have the speed with respect to a stationary reference. Thus:

$$\omega_r + \omega = \sigma\omega_s + \omega \qquad \text{(VII-3)}$$

Using (VII-1) in (VII-3) yields:

$$\omega_r + \omega = \left(\frac{\omega_s - \omega}{\omega_s}\right)\omega_s + \omega = \omega_s \qquad \text{(VII-4)}$$

Although ω_r = angular velocity of the rotor field with respect to the rotor, $\omega_r + \omega$ is the angular velocity of the rotor field with respect to a stationary reference. (VII-4) indicates that the speed of the rotor field with respect to the stationary reference is the same as that of the stator field. Therefore, the fields are stationary with respect to each other and an average toruqe is produced and the rotor continues to accelerate.

If we consider the situation as $\sigma \rightarrow 0$, i.e., the mechanical speed, ω, approaches ω_s we see that the relative velocity between stator field and rotor inductor approaches zero and under this condition, the induced rotor voltage also approaches zero. With zero voltage induced, no rotor current is possible and no torque can be produced. Thus, somewhere between $\sigma = 1.0$ (rotor stationary) and $\sigma = 0$ (synchronous speed) an equilibrium situation between developed torque and load torque is reached. The motor runs, steady state, at the speed at which this occurs. A typical speed torque characteristic is shown in Figure VII-1.

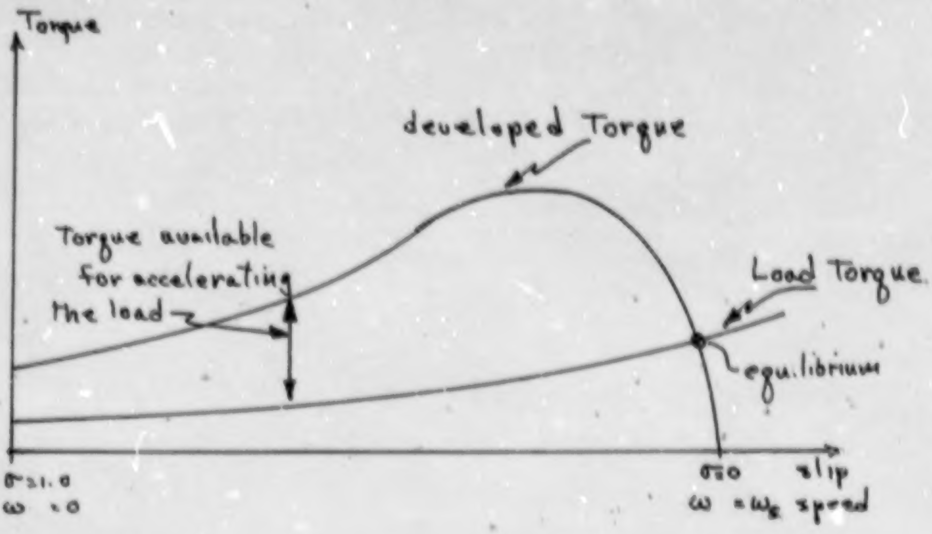
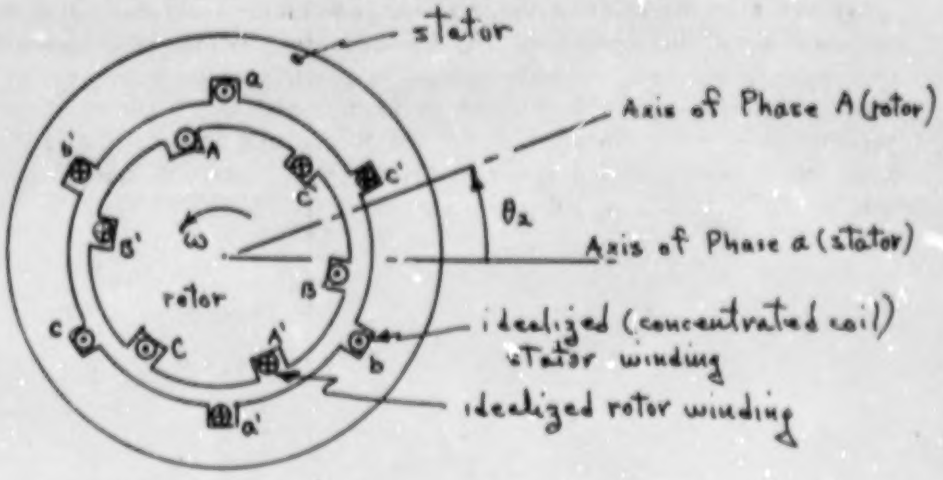


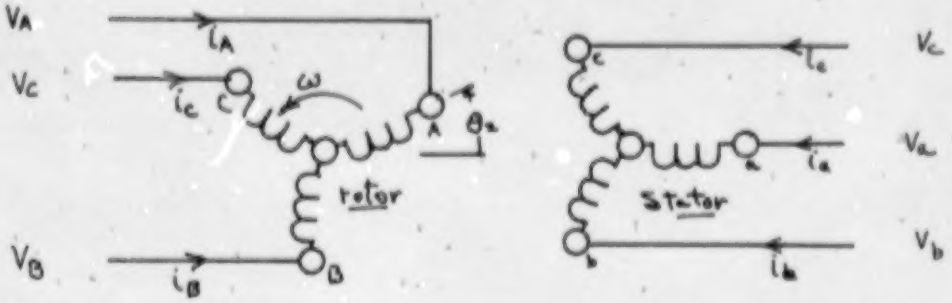
Figure VII-1. Induction Motor and Load Speed-Torque Characteristic

VII.2 INDUCTION MOTOR ANALYSIS: The 2 pole asynchronous machine is shown schematically in Figure VII-2. Note that the stator winding is depicted identical to the stator of the synchronous machine in Figure VI-2. Although the coils for phases a-b-c are shown concentrated they are usually distributed around the periphery in practice. The rotor winding is represented in the same fashion and the phases are denoted as A-B-C. The schematic representations for the machine are shown in Figure VII-2. The rotor windings are shown "open" rather than closed upon themselves and with voltages, v_A, v_B, and v_C impressed in order to obtain completely general results.

For analysis purposes, we can consider the instant of time when the axis of phase A on the rotor is at an angle θ_2 with respect to the axis of phase a on the stator. Since θ_2 is completely arbitrary, this is a generalized situation.


stator
a
b'
A
C
c'
Axis of Phase A (rotor)
θ_2
Axis of Phase a (stator)
ω
B'
B
rotor
c
C
A'
b
idealized (concentrated coil) stator winding
idealized rotor winding
a'

a) Induction Machine - Physical Layout


V_A
i_A
V_C
i_C
C
ω
A
θ_2
rotor
V_B
i_B
B
c
V_c
i_c
V_a
a
i_a
stator
b
V_b
i_b

b) Schematic

Figure VII-2. The 3 Phase Induction Machine

Now:

$$\theta_2 = \omega t \tag{VII-5}$$

or, since from (VII-1):

$$\omega = \omega_s (1 - \sigma)$$

$$\theta_2 = (1 - \sigma)\omega_s t \tag{VII-6}$$

If the machine has symmetrical windings (i.e. each stator phase is identical to the other stator phase winding except displaced in space, and the same holds true for the rotor windings) we can denote stator resistance as r_1, rotor resistance as r_2 and relate voltages, currents and flux linkages as follows:

$$v_a = r_1 i_a + \frac{d\lambda_a}{dt} \tag{VII-7}$$

$$v_b = r_1 i_b + \frac{d\lambda_b}{dt} \tag{VII-8}$$

$$v_c = r_1 i_c + \frac{d\lambda_c}{dt} \tag{VII-9}$$

$$v_A = r_2 i_A + \frac{d\lambda_A}{dt} \tag{VII-10}$$

$$v_B = r_2 i_B + \frac{d\lambda_B}{dt} \tag{VII-11}$$

$$v_C = r_2 i_C + \frac{d\lambda_C}{dt} \tag{VII-12}$$

The same assumptions made in connection with the analysis of the synchronous machine concerning saturation, hysteresis and eddy currents in the iron will be made here. In addition, as before, we will assume sinusoidal distribution of mmf and flux density in space. Further, if we neglect the permeance variations in the air gap due to the slots and teeth on the rotor and stator we note that all of the self inductances are independent of rotor position.

We will define the various self inductances as:

L_{aa}, L_{bb}, L_{cc} - stator winding self inductances

L_{AA}, L_{BB}, L_{CC} - rotor winding self inductances

For a symmetrical winding configuration

$$L_{aa} = L_{bb} = L_{cc} \tag{VII-13}$$

and

$$L_{AA} = L_{BB} = L_{CC} \tag{VII-14}$$

We will define the mutual inductances between stator windings as:

$$L_{ab} = L_{ac} = L_{bc} \tag{VII-15}$$

and the mutual inductances between rotor windings as

$$L_{AB} = L_{AC} = L_{BC} \tag{VII-16}$$

However, the mutual inductance of the rotor windings with respect to the stator windings (and vice versa) are functions of the rotor position which is a function of time. (The rotor position is defined for analysis purposes as θ_2).

The flux linkages in stator phase a due to rotor current i_A are a maximum when θ_2 is zero. With the assumption of sinusoidal flux density distributiuon they can be written as:

$$i_A L_{aA} \cos \theta_2 \tag{VII-17}$$

The flux linkages in stator phase a due to rotor current i_B are a maximum when $\theta_2 = 120^\circ$. Therefore they can be written as:

$$i_B \, L_{aA} \cos (\theta_2 - 120) \tag{VII-18}$$

and for rotor current i_C, the flux linkages in phase a can be expressed as

$$i_C \, L_{aA} \cos(\theta_2 - 240) \tag{VII-19}$$

Note that the peak mutual inductance between any stator and rotor winding is the same, denoted as L_{aA}, by virtue of symmetry.

Total flux linkages in phase a are therefore (including flux linkages due to stator currents)

$$\begin{aligned} \lambda_a = L_{aa} i_a + L_{ab} (i_b + i_c) + L_{aA} \, [i_a \cos \theta_2 \, + \\ i_B \cos (\theta_2 - 120) + i_C \cos (\theta_2 - 240)] \end{aligned} \tag{VII-20}$$

The flux linkages in rotor phase A due to various stator currents are:

$$i_a L_{aA} \cos \theta_2 \tag{VII-21}$$

$$i_b L_{aA} \cos (\theta_2 - 240) \tag{VII-22}$$

$$i_c L_{aA} \cos (\theta_2 - 120) \tag{VII-23}$$

Total flux linkages in rotor phase A are, therefore (including flux linkages due to rotor currents):

$$\lambda_A = L_{AA}i_A + L_{AB}(i_B + i_C) + L_{aA}[i_a \cos\theta_2 + i_b \cos(\theta_2 - 240) + i_c \cos(\theta_2 - 120)] \qquad \text{(VII-24)}$$

Note that the cosine expressions in λ_a and λ_A are not reciprocal inso far as the argument of the angles are concerned. This is because to an observer on the stator the rotor appears to rotate CCW where as to an observer on the rotor, the stator appears to rotate CW.

We can apply the same reasoning used above to write expressions for the flux linkages in the other windings. Thus

$$\lambda_b = L_{aa}i_b + L_{ab}(i_a + i_c) + L_{aA}[i_A \cos(\theta_2 - 240) + i_B \cos\theta_2 + i_c \cos(\theta_2 - 120)] \qquad \text{(VII-25)}$$

$$\lambda_B = L_{AA}i_B + L_{AB}(i_A + i_C) + L_{aA}[i_a \cos(\theta_2 - 120) + i_b \cos\theta_2 + i_c \cos(\theta_2 - 240)] \qquad \text{(VII-26)}$$

$$\lambda_c = L_{aa}i_c + L_{ab}(i_a + i_b) + L_{aA}[i_A \cos(\theta_2 - 120) + i_B \cos(\theta_2 - 240) + i_C \cos\theta_2)] \qquad \text{(VII-27)}$$

$$\lambda_C = L_{AA}i_C + L_{AB}(i_B + i_C) + L_{aA}[i_a \cos(\theta_2 - 240) + i_b \cos(\theta_2 - 120) + i_c \cos\theta_2] \qquad \text{(VII-28)}$$

Under balanced conditions, $i_a + i_b + i_c = 0$ and $i_A + i_B = i_C = 0$. This suggests rearrangement and regrouping of terms and the defining of fictitious inductnces, L_{11}, L_{22} as follows:

$$L_{aa}i_a + L_{ab}(i_b + i_c) = (L_{aa} - L_{ab})i_a = L_{11}\, i_a \qquad \text{(VII-29)}$$

and

$$L_{AA}i_A + L_{AB}(i_B + i_C) = (L_{AA} - L_{AB})\, i_A = L_{22}\, i_A \qquad \text{(VII-30)}$$

Using the defined inductances L_{11} and L_{22} we can summarize the flux linkage expressions above as:

$$\lambda_a = L_{11}i_a + L_{aA}\,[i_A \cos\theta_2 + i_B \cos(\theta_2 - 120) + i_C \cos(\theta_2 - 240)] \qquad \text{(VII-31)}$$

$$\lambda_b = L_{11}i_b + L_{aA}\,[i_A \cos(\theta_2 + 240) + i_B \cos\theta_2 + i_C\,(\theta_2 - 120)] \qquad \text{(VII-32)}$$

$$\lambda_c = L_{11}i_c + L_{aA}\,[i_A \cos(\theta_2 + 120) + i_B \cos(\theta_2 - 240) + i_C \cos\theta_2] \qquad \text{(VII-33)}$$

$$\lambda_A = L_{22}i_A + L_{aA}[i_a \cos\theta_2 + i_b \cos(\theta_2 - 240) + i_c \cos(\theta_2 - 120) \qquad \text{(VII-34)}$$

$$\lambda_B = L_{22}i_B + L_{aA}\,[i_a \cos(\theta_2 - 120) + i_b \cos\theta_2 + i_c \cos(\theta_2 - 240) \qquad \text{(VII-35)}$$

$$\lambda_C = L_{22}i_C + L_{aA}\,[i_a \cos(\theta_2 - 240) + i_b \cos(\theta_2 - 120) + i_c \cos\theta_2] \qquad \text{(VII-36)}$$

If the flux linkages, as expressed in (VII-31) through (VII-36) are used in the voltage equations (VII-7) through (VII-12) a set of non linear differential equations result because both currents and angle are functions of time. The bracketed coefficient of L_{aA} in (VII-31) through (VII-37) suggests the use of the same type of transformation as used in the synchronous machine analysis (refer to Section VI.3). Recall in that analysis the direct axis was chosen along the rotor axis with the quadrature axis advanced 90° ahead in the direction of rotation. The transformation developed had the mathematical effect of converting the three stationary phase windings (and current, voltage, and flux linkages) to two fictitious windings denoted as d and q windings-one aligned along the direct axis and the other aligned along the quadrature axis.

The fictitious d and q windings rotate at synchronous speed and thus are stationary with respect to the stator. The fact that the actual windings were rotating with respect to air gap flux density is accounted for by virtue of the $\omega\lambda_q$ and $\omega\lambda_d$ speed voltages in the analysis.

For the induction machine we can use the same approach but we will have to use a transformation on both the stator and rotor winding because both windings rotate relative to the stator field. We will choose the diredt axis such that it rotates synchronously with the stator field and locate it so that it passes through the a phase winding axis at $t = 0$. From (VI-34,-35) we express the direct and quadrature axis stator currents as: (using the additional subscript, 1, to denote stator quantities)

$$\begin{vmatrix} i_{d1} \\ i_{q1} \\ i_{o1} \end{vmatrix} = \frac{2}{3} \begin{vmatrix} \cos\theta & \cos(\theta - 120) & \cos(\theta + 120) \\ -\sin\theta & -\sin(\theta - 120) & -\sin(\theta + 120) \\ 1/2 & 1/2 & 1/2 \end{vmatrix} \bullet \begin{vmatrix} i_a \\ i_b \\ i_c \end{vmatrix} \qquad \text{(VII-37)}$$

and, from the inverse of (VII-37)

$$\begin{vmatrix} i_a \\ i_b \\ i_c \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \cos(\theta - 120) & -\sin(\theta - 120) & 1 \\ \cos(\theta + 120) & -\sin(\theta + 120) & 1 \end{vmatrix} \begin{vmatrix} i_{d1} \\ i_{q1} \\ i_{o1} \end{vmatrix} \tag{VII-38}$$

where $\theta = \omega_s t$ and ω_s is the angular velocity of the stator field.

The rotor windings are at a different angle with respect to the d-q axis than are the stator windings so the transformation used for the rotor windings will have a different angle than $\theta = \omega_s t$. θ_2 was previously defined as the angle of rotor phase A ahead of stator phase a winding axis. Thus, the angle between phase A axis and the d axis, which we will define as θ_r is the difference between $\theta = \omega_s t$ and θ_2. Or,

$$\theta_r = \omega_s t - \theta_2; \; \theta_2 = \theta - \theta_r \tag{VII-39}$$

In the transformation (VII-37,-38) the angle between the a phase winding axis and the d axis was used. For the rotor quantities we would use the same transformation except with θ replaced by θ_r, the angle between the d axis and the axis of rotor phase A and we will use the additional subscript 2 to denote rotor quantities. As before, the transformations are valid for voltages, current and flux linkages.

If the transformations are used with equations (VII-7) through (VII-12) and (VII-31) through (VII-36) the following results are obtained. (Balanced voltages are assumed so the zero sequence quantities are non existent).

$$v_{1d} = r_1 i_{1d} + \frac{d\lambda_{1d}}{dt} - \omega\lambda_{1q} \tag{VII-40}$$

$$v_{1q} = r_1 i_{1q} + \frac{d\lambda_{1q}}{dt} + \omega_s \lambda_{1d} \tag{VII-41}$$

$$v_{2d} = r_2 i_{2d} + \frac{d\lambda_{2d}}{dt} - \lambda_{2q} \frac{d\theta_r}{dt} \tag{VII-42}$$

$$v_{2q} = r_2 i_{2q} + \frac{d\lambda_{2q}}{dt} + \lambda_{2d} \frac{d\theta_r}{dt} \tag{VII-43}$$

where:

$$\lambda_{id} = L_{11} i_{1d} + \frac{3}{2} L_{aA}\, i_{2d} \qquad \text{(VII-44)}$$

$$\lambda_{1q} = L_{11} i_{2q} + \frac{3}{2} L_{aA}\, i_{2q} \qquad \text{(VII-45)}$$

$$\lambda_{2d} = L_{22} i_{2d} + \frac{3}{2} L_{aA}\, i_{1d} \qquad \text{(VII-46)}$$

$$\lambda_{2q} = L_{22} i_{2q} + \frac{3}{2} L_{aA}\, i_{1q} \qquad \text{(VII-47)}$$

Recall that θ_r is the angle between the d axis, rotating at synchronous speed and the rotor, rotating at speed ω. Therefore $\frac{d\theta_r}{dt}$ is the relative speed between the stator field and the rotor. From the definition of slip, :

$$\sigma\omega_s = \omega_s - \omega \qquad \text{(VII-1)}$$

we note that:

$$a\omega_s = \frac{d\theta_r}{dt} \qquad \text{(VII-48)}$$

The equations above for rotor and stator relationships are compatible with physical reasoning for the rotor and stator circuits. The stator equations are of the same form as those of the synchronous machine stator if we recognize that L_{11} and L_{22} are self inductances and $\frac{3}{2} L_{aA}$ is the mutual inductance between rotor and stator. For the rotor, the equations are also of the same form. For the stator phase windings, the speed voltages are proportional to ω_s. For the rotor windings, the speed voltages are proportional to $\sigma\omega_s$ because that is the magnitude of the speed of the d axis relative to the rotor winding.

The instantaneous power associated with the polyphase machine is the sum of the products of voltage and current in each winding - either using phase quantities or the transformed d-q quantities. In connection with the generalized machine analysis it was demonstrated that three type terms results, i.e.,

i^2r - heat loss

$i\,\frac{d\lambda}{dt}$ - change of stored energy

$\omega i \lambda$ - power associated with speed voltages

It was demonstrated that only the power terms associated with speed voltages represent power converted in the electromechanical conversion process. Since the electromechanical energy conversion process takes place in the rotor, we can focus our attention on terms involving rotor speed voltages and currents to obtain the expression for converted power, P_2, associated with the induction machine. The result is

$$P_2 = \frac{3}{2} \omega_s \sigma [\lambda_{2d} i_{2q} - \lambda_{2q} i_{2d}] \qquad \text{(VII-49)}$$

Now, $-\omega_s \sigma$ is the rotor speed with respect to the d-q axis in electrical radian/sec, because the angle θ_r is constantly increasing in a negative direction $(\omega_s > \omega)$. The speed in mechanical radians per second is, then:

$$\omega = -\frac{\omega_s \sigma}{P/2} \qquad \text{(VII-50)}$$

where P is the number of poles. Since:

$$T_e = \frac{P_2}{\omega} = \frac{3}{2}\,\frac{P}{2}\,(\lambda_{2q}\, i_{2d} - \lambda_{2d}\, i_{2q}) \qquad \text{(VII-51)}$$

Of course, the usual relationship between the torque produced, the load torque and the inertia torque also exists, i.e.

$$T_e = T_L + J\,\frac{d\omega}{dt} \qquad \text{(VII-52)}$$

In the equation developed, positive quantities pertain to motor action. If, for example, ω, was greater than ω_s (only possible if T_L is negative indicating load torque into the machine - or, in other words, the induction machine is being driven and not functioning as the driver) then slip, σ, is negative and mechanical power is converted to electrical form and the machine is functioning as a generator.

We can summarize the equations developed for the induction machine as follows:

$$\sigma = \frac{\omega_s - \omega}{\omega} \qquad \text{(VII-53)}$$

(ω_s and ω must both be in either electrical or mechanical units)

$$v_{1d} = r_1 i_{1d} + \frac{d\lambda_{1d}}{dt} - \omega\lambda_{1q} \qquad \text{(VII-54)}$$

$$v_{1q} = r_1 i_{1q} + \frac{d\lambda_{1q}}{dt} + \omega\lambda_{id} \qquad \text{(VII-55)}$$

$$v_{2d} = r_2 i_{2d} + \frac{d\lambda_{2d}}{dt} - \sigma\omega\lambda_{2q} \qquad \text{(VII-56)}$$

$$v_{2q} = r_q i_{2q} + \frac{d\lambda_{2q}}{dt} + \sigma\omega\lambda_{2d} \qquad \text{(VII-57)}$$

$$\lambda_{1d} = L_{11} i_{1d} + L_{12} i_{2d} \qquad \text{(VII-58)}$$

$$\lambda_{1q} = L_{11} i_{1q} + L_{12} i_{2q} \qquad \text{(VII-59)}$$

$$\lambda_{2d} = L_{22} i_{2d} + L_{12} i_{1d} \qquad \text{(VII-60)}$$

$$\lambda_{2q} = L_{22} i_{2q} + L_{i2} i_{1q} \qquad \text{(VII-61)}$$

where

r_1 = stator resistance/phase

r_2 = rotor resistance/phase

$L_{11} = L_{aa} - L_{ab}$

$L_{22} = L_{AA} - L_{AB}$

$L_{12} = \frac{3}{2} L_{aA}$

L_{aa} = stator phase self inductance

L_{AA} = rotor phase self inductance

L_{ab} = mutual inductance between two stator phases

L_{AB} = mutual inductance between two rotor phases

L_{aA} = maximum value of mutual inductance between a stator phase and a rotor phase (when the two phase windings are aligned)

$$T_e = \frac{3}{2}\left(\frac{P}{2}\right)(\lambda_{2q}\, i_{2d} - \lambda_{2d}\, i_{2q}) \qquad \text{(VII-62)}$$

where

P = number of poles

$$T_e = T_L + J \frac{d\omega}{dt} \qquad \text{(VII-63)}$$

where

ω = mechanical speed of rotor

J = total moment of inertia on shaft

T_e = torque associated with the conversion process

T_L = torque to connected load. Positive for a driven load (motor)

We are now in a position to use the equations which describe the induction machine to analyze the machine behavior under various conditions and situations.

VII.3 STEADY STATE REPRESENTATION: Under steady state conditions, i.e. constant slip, the transformed windings are stationary with respect to each other and the flux linkages are constant. Therefore the terms involving derivatives of flux linkages are zero.

Since

$$v_a = v_{1d} \cos \omega_s t - v_{1q} \sin \omega_s t \qquad \text{(VII-64)}$$

$$= v_{1d} \cos \omega_s t + v_{1q} \cos (\omega_s t + 90) \qquad \text{(VII-65)}$$

we can write

$$v_1 = v_{1d} + j\, v_{1q} \qquad \text{(VII-66)}$$

Similarly;

$$i_1 = i_{1d} + j\, i_{1q} \quad \text{for the stator} \qquad \text{(VII-67)}$$

and

$$i_2 = i_{2d} + j\, i_{2q} \quad \text{for the rotor} \qquad \text{(VII-68)}$$

where v_1 is the instantaneous value of line to neutral (phase) voltage impressed on the stator and i_1 and i_2 are the instantaneous stator and rotor currents. In the normal mode of operation, the rotor windings are closed upon themselves. Thus, with no externally applied voltages:

$$v_{2d},\ v_{2q} \text{ are zero} \qquad \text{(VII-69)}$$

In the equations above, L_{11} and L_{22} are self inductances whereas L_{12} is the mutual inductance between stator and rotor. We will introduce the leakage reactances L_1 and L_2 (of the stator and rotor respectively). Thus:

$$L_{11} - L_1 = L_{12} \tag{VII-70}$$

$$L_{22} - L_2 = L_{12} \tag{VII-71}$$

For the induction machine in the steady state, using (VII-70,-71) we can write the describing equations, from (VII-54 through -61) as follows:

$$v_{1d} = r_1 i_{1d} - \omega_s (L_{12} + L_1)\, i_{1q} - \omega_s L_{12}\, i_{2q} \tag{VII-72}$$

which can be written, for steady state sinusoidal conditions:

$$v_{id} = r_1 i_{1d} + J\, \omega_s (L_{12} + L_1)(j\, i_{2q}) + j\, \omega_s L_{12}\, (j\, i_{2q}) \tag{VII-73}$$

and

$$v_{1q} = r_1 i_{1q} + \omega_s (L_{12} + L_1) i_{1d} + \omega_s L_{12} i_{2d} \tag{VII-74}$$

or

$$j\, v_{1q} = r_1 (j\, i_{1q}) + j\, \omega_s (L_{12} + L_1) i_{1d} + j\, \omega_s L_{12} i_{2d} \tag{VII-75}$$

Adding (VII-73) and (VII-75) yields:

$$v_{1d} + j\, v_{1q} = r_1 (i_{1d} + j\, i_{1q}) + (i_{1d} + j\, i_{1q})(j\, \omega_s L_1) +$$

$$(i_{1d} + j\, i_{1q})(j\, \omega_s L_{12}) + (i_{2d} + j\, i_{2q})(j \omega_s L_{12}) \tag{VII-76}$$

Using the value of i_1, i_2 and v_1 from (VII-66,-67 and -68) enables (VII-76) to be written as:

$$v_1 = r_1 i_1 + i_1\, (j\, \omega_s L_1) + (i_1 + i_2)(j\, \omega\, L_{12}) \tag{VII-77}$$

Equation (VII-77) expresses a relationship between instantaneous values. If each term were divided by $\sqrt{2}$ it would express the relationship in terms of rms values; thus,

$$V_1 = r_1 I_1 + j\, I_1 X_1 + j\, (I_1 + I_2)\, X_{12} \tag{VII-78}$$

where:

V_1 = rms applied voltage per phase

I_1, I_2 rms stator and rotor currents

X_1 = stator leakage reactance = $\omega_s L_1$

X_{12} = stator to rotor mutual reactance = $\omega_s L_{12}$

We next apply the same procedure to the equations involving the rotor circuits. Thus:

$$0 = r_2 i_{2d} - \sigma\omega_s(L_{12} + L_2)\, i_{2q} - \sigma\omega_s L_{12}\, i_{1q} \qquad \text{(VII-79)}$$

or:

$$0 = r_2 i_{2d} + \sigma\,(j\omega_s)(L_{12} + L_2)(j\, i_{2q}) + \sigma\,(j\omega_s)\, L_{12}(j\, i_{1q}) \qquad \text{(VII-80)}$$

and:

$$0 = r_2 i_{2q} + \sigma\omega_s(L_{12} + L_2)\, i_{2d} + \sigma\omega_s L_{12}\, i_{1d} \qquad \text{(VII-81)}$$

or:

$$0 = r_2\,(j\, i_{2q}) + \sigma(j\omega_s)(L_{12} + L_2)\, i_{2d} + \sigma\,(j\omega_s)\, L_{12}\, i_{1d} \qquad \text{(VII-82)}$$

adding (VII-80) to (VII-82) yields:

$$0 = r_2(i_{2d} + j\, i_{2q}) + j\,(\sigma\omega_s L_{12})(i_{1d} + j\, i_{1q} + i_{2d} + j\, i_{2q}) + j(\sigma\omega L_2)(i_{2d} + j\, i_{2q}) \qquad \text{(VII-83)}$$

Converting to rms values, and using (VII-67,-68), we have:

$$0 = r_2 I_2 + j(\sigma X_{12})(I_1 + I_2) + j\sigma X_2 I_2 \qquad \text{(VII-84)}$$

where X_2 = rotor leakage reactance at stator frequency = $\omega_s L_2$.

The coefficient σ in front of X_{12}, X_2 indicates that the actual rotor frequency is stator frequency times slip. This is reasonable since the relative speed between the rotor and the stator field is $\sigma\omega_s$.

Equivalent circuits are very useful because we can then utilize conventional circuit theory techniques for behavior analysis. We can formulate equivalent circuits shown in Figure VII-3 for (VII-78) and (VII-84) if we divide the latter by slip, σ.

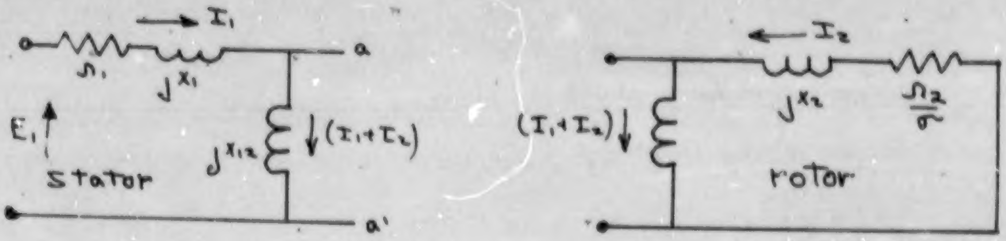


Figure VII-3. Steady State Equivalent Circuits - Asynchronous Machines

Since these circuits have a common element, jX_{12} they can be combined into a single circuit as shown in Figure VII-4. The total power absorbed in the rotor, per phase is:

$$I_2^2 \frac{r_2}{\sigma} \tag{VII-85}$$

Some of this power, i.e.,

$$I_2^2 r_2 \tag{VII-86}$$

is dissipated as rotor copper loss. The remainder,

$$I_2^2 \frac{r_2}{\sigma} - I^2 r_2 = I_2^2 r_2 \left(\frac{1-\sigma}{\sigma}\right) = P_o \tag{VII-87}$$

is converted to mechanical form and is the shaft power output plus the power loss associated with mechanical friction and windage. This suggests splitting r_2/σ, the rotor equivalent circuit resistance, into two rotor resistors,

$$r_2 \text{ and } r_2 \left(\frac{1-\sigma}{\sigma}\right) \tag{VII-88}$$

to represent the absorption of rotor power.

The equivalent circuit is shown in Figure VII-4.

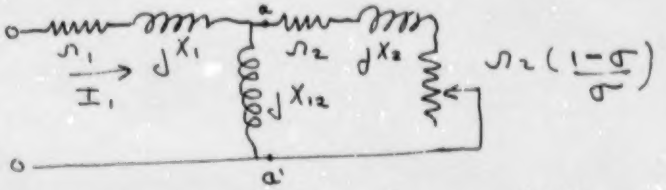


Figure VII-4. The Asynchronous Machine Equivalent Circuit (One Phase Only)

Another technique which will be useful is to find the Thevenin equivalent circuit for the stator. To do so, find the open circuit, i.e., rotor removed, voltage across a-a' in Figure VII-4. It is

$$j\, I_1 X_{12} = \frac{V_1(j\, X_{12})}{r_1 + j(X_1 + X_{12})} = V_1' \tag{VII-89}$$

In a well designed machine, $(X_{12} + X_1) \gg r_1$ and

$$V_1' \approx \frac{X_{12}\, E_1}{X_1 + X_{12}} \tag{VII-90}$$

To find the equivalent series impedance, $R_1 + j\, X_1$, for the Thevenin circuit representation, measure the impedance between a-a', with V_1 replaced by a zero impedance connection.

$$R_1 + j\, X_1 = \frac{(r_1 + j\, x_1)\, j\, x_{12}}{r_1 + j\,(x_1 + x_{12})} \approx \frac{x_{12}}{x_1 + x_{12}}\,(r_1 + j\, x_1) \tag{VII-91}$$

Using the values from (VII-90,-91) and with the alteration in r_2/σ discussed above, we arrive at the final version of the circuit representation of one phase of the asynchronous induction machine. It is shown in Figure VII-5.

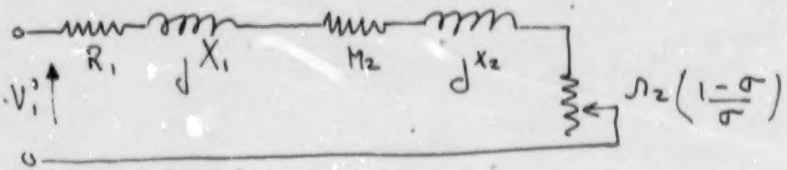


Figure VII-5. Steady State Thevenin Equivalent Circuit

The resistor representing power converted to mechanical form is shown variable because its value is dependent upon slip, σ. Since we have assumed balanced and symmetrical phase, whatever we calculate for this phase is also happening in the other phases! Thus, quantities such as power loss, power converted, etc. must be multiplied by three (in a three phase machine) to account for all three phases!

VII.4 STEADY STATE PERFORMANCE: For a motor, in the steady state, the performance quantities of interest are usually the speed torque characteristic, the current and the efficiency for various operating speeds. These quantities can be readily determined by use of the equivalent circuit depicted in Figure VII-5.

The power, in the three phases, converted to mechanical form is:

$$P_o = 3\, I_2^2 r_2 \left(\frac{1-\sigma}{\sigma}\right) \qquad \text{(VII-92)}$$

The torque developed is:

$$T_e = \frac{P_o}{\omega} = \frac{P_o}{\omega_s (1-\sigma)} = \frac{3\, I_2^2\, r_2 (1-\sigma)}{2\pi f\, (1-\sigma)\sigma} \left(\frac{P}{2}\right) \qquad \text{(VII-93)}$$

Since:

$$f = \left(\frac{\omega}{2\pi}\right)\left(\frac{P}{2}\right) \qquad \text{(VII-94)}$$

From the equivalent circuit:

$$|I_2^2| = \frac{V_1^2}{\left(R_1 + \frac{r_2}{\sigma}\right)^2 + \left(X_1 + x_2\right)^2} \qquad \text{(VII-95)}$$

$$\therefore\ T_e = \frac{3}{2\pi f}\left(\frac{P}{2}\right) \frac{V_1^2\, r_2/\sigma}{\left(R_1 + \frac{r_2}{\sigma}\right)^2 + \left(X_1 + x_2\right)^2} \qquad \text{(VII-96)}$$

T_e as a function o f σ, and also ω is plotted in Figure VII-6. Note that σ can vary over a range from $\sigma < 0$ through $\sigma = 0$ and then into the region $\sigma > 0$. $\sigma > 0$ represents motor action; $\sigma > 0$ corresponds to generator action. $\sigma = 0$ is the transition point. The maximum torque developed is designated as T_m and it occurs at a slip σ_m.

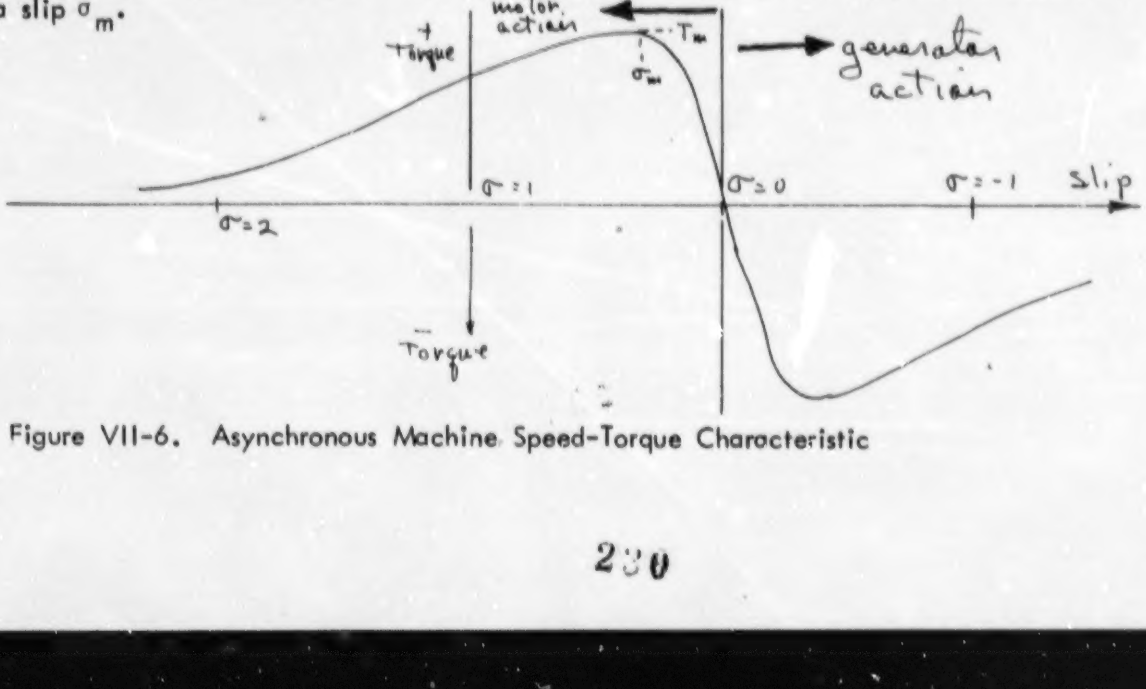


Figure VII-6. Asynchronous Machine Speed-Torque Characteristic

σ_m is determined by equating the derivative of T_e with respect to σ equal to zero. Thus, from (VII-96)

$$\frac{d\,T_e}{d\sigma} = \frac{\frac{3}{2\pi f}\left(\frac{P}{2}\right) V_1'^2 r_2' \left\{\left[\left(R_1 + \frac{r_2}{\sigma}\right)^2 + (X_1 + x_2)^2\right]\left(-\frac{1}{\sigma^2}\right) - \left(\frac{1}{\sigma}\right)\left[2\left(R_1 + \frac{r_2}{\sigma}\right)\left(-\frac{r_2}{\sigma^2}\right)\right]\right\}}{\left[\left(R_1 + \frac{r_2}{\sigma}\right)^2 + (X_1 + x_2)^2\right]^2} = 0 \qquad \text{(VII-97)}$$

(VII-97) is solved for the σ which yields T_m which we designate σ_m. Thus:

$$\sigma_m = \frac{r_2}{\sqrt{R_1^2 + (X_1 + x_2)^2}} \qquad \text{(VII-98)}$$

If we substitute (VII-98) into the expression for T_e (VII-96) we will have T_m. Thus:

$$T_m = \left(\frac{3}{2}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{P}{2}\right) \frac{V_1'^2}{R_1 + \sqrt{R_1^2 + (X_1 + x_2)^2}} \qquad \text{(VII-99)}$$

At this point we can make several very important observations regarding the speed-torque characteristic of an induction motor, i.e.,

(1) From (VII-96), developed torque is proportional to the square of the Thevenin circuit voltage and thus approximately proportional to the square of the applied voltage.

(2) From (VII-99) the maximum torque developed is independent of rotor resistance, r_2.

(3) From (VII-98), the slip, σ_m, at which maximum torque is developed is directly proportional to r_2, (even though T_m itself is independent of r_2).

If we are dealing with a wound rotor machine we can add external resistance at the slip rings in order to increase r_2 above the value inherent in the actual rotor winding. For this configuation, the speed-torque characteristic is actually a family of curves, as shown in Figure (VII-7). (Only the portion corresponding to motoring action is shown).

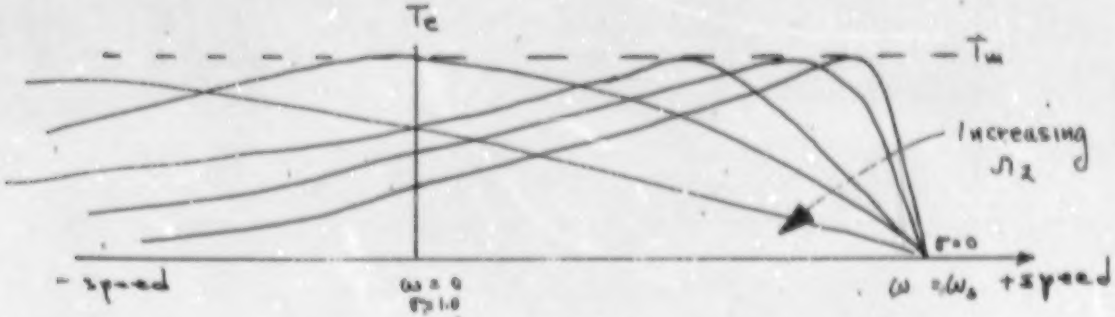


Figure VII-7. Speed-Torque Characteristics Indicating Effect of Increasing Rotor Resistance

The capability of varying the speed torque characteristic by insertion of external resistance in the rotor circuit is the justification for use of the wound rotor motor instead of the squirrel cage type. However, increased mechanical complexity associated with slip rings, an actual winding, etc. result in a design that is more expensive than the simple squirrel cage configuration.

It is possible by the proper choice of external resistance to cause the motor to develop maximum torque at $\sigma_m = 1.0$, i.e., on starting. Wound rotor motors are sometimes specified for this purpose. More often they are utilized to increase circuit impedance and limit inrush current on starting, when r_2/σ is a minimum and consequently when total circuit impedance is at its lowest value. The current for any value of slip, σ, can be calculated from the equivalent circuit. Since $(X_1 + x_2) >> R_1 + \frac{r_2}{\sigma}$ when $\sigma = 1$, the inrush current on start is very nearly in quadrature with the voltage. In a typical squirrel cage machine its value will be 5-7 times that of full load running current. Since this may result in underirable voltage fluctuations in the supply circuit, either additional rotor resistance or starting on reduced voltage may be required. Section VII.8 deals with starting problems.

The variable speed torque characteristic of the wound rotor motor can be used to obtain a variable speed drive capability from a machine energized from a constant voltage-constant frequency alternating current system. This useage was greater before the advent of power semiconductors capable of supplying d.c. for a shunt d.c. motor or for obtaining variable frequency a.c. drive. This will be discussed in Section VII.7. The disadvantage of operating an induction motor at high values of slip can be visualized when one considers that the power, P_g, transferred to the rotor from the stator is:

$$P_g = 3\, I_2^2 \frac{r_2}{\sigma} \qquad \text{(VII-100)}$$

The power output, P_o, is, from (VII-92)

$$P_o = 3\, I_2^2\, r_2 \left(\frac{1-\sigma}{\sigma}\right) \qquad \text{(VII-92)}$$

The ratio of P_o/P_g is a measure of the efficiency of the rotor circuit. It is:

$$\frac{P_o}{P_g} = 1 - \sigma \qquad \text{(VII-101)}$$

or

$$P_o = (1 - \sigma)\, P_g \qquad \text{(VII-102)}$$

One can readily see that operation at high slip is inherently inefficient and may be quite costly for motors of large size. Indeed, there have been several schemes utilized to convert this rotor power back into power at stator frequency and pump it back into the stator supply circuit. These schemes will not be discussed here but descriptions of them are available in the literature. In addition to the economic penalties associated with operation at high slip one must consider the thermal limitations that arise.

The rotor copper loss, P_{cu}, is the difference between P_g and P_o. Thus

$$P_{cu} = P_g - P_o \qquad \text{(VII-103)}$$

Using (VII-102), we find P_{cu} as a function of power across the air gap and slip as:

$$P_{cu} = \sigma\, P_g \qquad \text{(VII-104)}$$

Most motors rely on air forced through the air gap by fan blades located on the rotor to remove heat from the rotor. At high slip, more losses exist in the rotor but the rotor is running at reduced speed and is less capable of heat removal thanit is when rotor losses are less! Most motors for variable speed operation will be derated as far as power output is concerned when operated at reduced speeds.

To calculate efficiency, we must first categorize the losses. These losses are added to the output to obtain the input. The efficiency is then calculated as:

$$\%\ \text{efficiency} = \frac{P_{out}}{P_{in}} \times 100 = \left(\frac{P_{in} - \text{losses}}{P_{in}}\right) 100$$

$$= 1 - \frac{\text{losses}}{P_{out} + \text{losses}}) \, 100 \qquad \text{(VII-105)}$$

In general, there are four losses to consider:

1) the i^2r, or copper losses. These are self explanatory and are given as $3\ I_2^2(R_1 + r_2)$ for the machine.

2) mechanical losses are those associated with mechanical friction (bearings) and windage loss.

3) no load (core) losses. These losses arise from the time varying flux densities in the iron. They are dependent upon the maximum value of flux density and are substantially constant when the impressed voltage and frequency are constant.

4) stray load losses. These losses result from non-uniform current distribution in the conductors and additional core loss produced in the iron by distortion of flux due to load current.

Oft times the no load and stray load losses are lumped together and classified as "electrical constant" losses. For normal operation of the machine near synchronous speed (low slips) the mechanical losses are also considered constant although they vary with the 2nd or 3rd power of speed in practice. If we define:

P_s = stray load losses

P_c = core losses

P_m = mechanical losses

P_o = power converted to mechanical form $= 3\ I_2^2 r_2 (\frac{1-\sigma}{\sigma})$
(this includes the mechanical losses)

and

P_{cu} = copper losses $= 3\ I_2^2\ (R_1 + r_2)$

we can calculate efficiency, η, as follows:

$$\eta = \left(1 - \frac{P_s + P_c + P_m + P_{cu}}{P_o + P_s + P_c + P_{cu} + P_m}\right)100 \qquad \text{(VII-106)}$$

VII.5 AN APPROXIMATE EXPRESSION FOR TORQUE: The expression for torque (VII-96) requires a knowledge of the machine parameters which can be obtained by farily straight forward test procedures. However, the size of the machine, or the fact that the machine is not available for test at the time of an analysis make preclude such tests. An approximation to (VII-96) has been found to be very useful and is based on data which can be estimated rather easily.

If we assume that $R_1 \ll (X_1 + x_2)$ and neglect R_1, (VII-98) becomes

$$\sigma_m = \frac{r_2}{X_1 + x_2} \qquad \text{(VII-107)}$$

and, from (VII-99):

$$T_m = \frac{3}{2\omega} \; \frac{V_1'^2}{(X_1 + x_2)} \qquad \text{(VII-108)}$$

from (VII-96):

$$T_e = \frac{3\,V_1^2}{\omega} \; \frac{r_2/\sigma}{\left(\frac{r_2}{\sigma}\right)^2 + (X_1 + x_2)^2} \qquad \text{(VII-109)}$$

Substituting $X_1 + x_2 = \frac{r_2}{\sigma_m}$ from (VII-107) into (VII-108,-109), and dividing (VII-109) by (VII-108) yields:

$$T_e = \frac{2\,T_m}{\frac{\sigma}{\sigma_m} + \frac{\sigma_m}{\sigma}} \qquad \text{(VII-110)}$$

Thus if we know the maximum (also called breakdown) torque and the slip at which it occurs we can obtain an approximate mathematical expression for T_e as a function of slip, σ.

VII.6 MACHINE PARAMETERS BY TEST: The IEEE Test Code, for polyphase induction motors, #500, gives complete information concerning induction motor test procedures. This discussion will limit itself to the basic concepts of impedance determination.

The stator resistance, r_1, can be measured by regular iR drop techniques using direct currents. Then, if power, voltage and current are measured at no load we can determine $X_{12} + X_1$ as follows:

Let $P_{n\ell}$, $I_{n\ell}$, $V_{n\ell}$ be the no load power, current and voltage. At no load, σ is very low and $\frac{r_2}{\sigma}$ is a large number, approaching ∞ as $\sigma \rightarrow 0$. If $\sigma = 0$, the branch containing X_{12} is shunted by an infinite impedance and the impedance of the parallel combination approaches $j\,X_{12}$. We can calculate:

$$R_{n\ell} = \frac{P_{n\ell}}{3\,I_{n\ell}^2} \quad \text{and} \quad Z_{n\ell} = \frac{V_{n\ell}/\sqrt{3}}{I_{n\ell}} \qquad \text{(VII-111)}$$

from which:

$$X_{n\ell} = \sqrt{Z_{n\ell}^{\,2} - R_{n\ell}^{\,2}} \qquad \text{(VII-112)}$$

$X_{n\ell}$ as calculated from (VII-112) is very nearly:

$$X_{n\ell} = x_1 + x_{12} \qquad \text{(VII-113)}$$

Now, if the rotor is blocked so that it cannot turn, and $P_{B\ell}$, $V_{B\ell}$, $I_{B\ell}$ are power, voltage and current readings taken at the terminals:

$$R_{B\ell} = \frac{P_{B\ell}}{3\, I_{B\ell}^{\,2}} \quad \text{and} \quad Z_{B\ell} = \frac{V_B/\sqrt{3}}{I_{B\ell}} \qquad \text{(VII-114)}$$

from which:

$$X_{B\ell} = \sqrt{Z_{B\ell}^{\,2} - R_{B\ell}^{\,2}} \qquad \text{(VII-115)}$$

This test is made on reduced voltage (approximately 20% of rated) so that I_B is near normal full load current. Now the branch, $j\, x_2 + r_2/\sigma = j\, x_2 + r_2$ (since $\sigma = 1$) is shunted by X_{12}. However, $j\, X_{12} >> j\, x_2 + r_2$ and we will assume that the impedance of the parallel combination approaches the impedance $r_2 + j\, x_2$. This test is similar to the "short circuit" test on a transformer. With the assumptions made:

$$r_1 + r_2 = \frac{P_{B\ell}}{3\, I_{B\ell}^{\,2}} \quad \text{and} \quad Z_{B\ell} = \frac{V_{B\ell}}{3\, I_{B\ell}} \qquad \text{(VII-116)}$$

from which:

$$x_1 + x_2 = \sqrt{Z_{B\ell}^{\,2} - (r_1 + r_2)^2} \qquad \text{(VII-117)}$$

and

$$r_2 = \frac{P_{B\ell}}{3\, I_{B\ell}^{\,2}} - r_1 \qquad \text{(VII-118)}$$

At this point, an estimate of the ratio x_1/x_2 must be made. There are empirical values of this ratio available for different motor designs. For example, wound rotor motors and cage motors designed for normal starting torque-normal starting current, and cage type motors designed for high starting torque-high running slip have an empirical ratio $x_1/x_2 = 1.0$. On the other hand, the class of motor designed for normal starting torque-low starting current has a ratio $x_1/x_2 = .67$. From this ratio and (VII-118), a value for x_1 can be obtained. The value of x_1 and (VII-113) enables us to determine x_{12}. Thus, all parameters can be obtained. It should be emphasized that the test procedures given in IEEE #500 should be followed for accurate results. The procedures presented here are to yield insight only. They are relatively unsophisticated.

Typical motors have maximum torque values between 200 and 350% of rated torque values. Except in the case of "high slip" motors, this maximum torque usually occurs at a slip of 8-12%. "High slip" motors usually have a maximum torque in excess of 300% with the maximum value occurring at slips in excess of 60%. Starting torques range from 150-300% of rated torque values (for starting at rated voltage). The high slip motor usually has larger values of starting torque than conventional slip motors.

VII.7 ASYNCHRONOUS GENERATORS: Note from Equation (VII-96) that if the slip is negative the numerator changes sign whereas the denominator retains the positive sign because of the presence of slip in squared form. This means torque becomes negative and corresponds to generator action on the part of the asynchronous machine. This is shown in graphic form in Figure VII-6. Physically the negative slip, i.e., slip less than 0, corresponds to a rotor speed greater than synchronous speed which means practically that the rotor is being driven so that it runs ahead of the synchronously rotating magnetic field. It should be noted that the rotating magnetic field in the polyphase induction motor owes its existence to the current supplied to the stator winding from the line and that there is no other source of excitation in the machine. This means that if we are to operate the induction machine as an asynchronous generator this supply of magnetizing current must continue to be available to the machine after the speed has passed beyond synchronous speed and the machine is in a generating mode. In other words, a generator of this type is not self-exciting but must be operated in parallel with other generators, an electrical system, or static capacitors which are capable of supplying the machine with exciting current. If the asynchronous generator is operated in parallel with another system, or other generators, they can supply it with the necessary excitation at fixed frequency.

So long as the exciting current is supplied at fixed frequency (through the stator winding) the frequency relationship between rotor winding and the mechanical speed of the rotor winding will adjust to insure that power at the supply frequency is maintained and supplied back to the stator. Thus the asynchronous generator can supply fixed frequency power back into the system regardless of the speed of the synchronous generator rotor. Practically, the range over which an asynchronous generator can be operated varies from a slip of zero up to probably five to eight percent negative. This in turn corresponds to very nearly rated torque of the machine and does not pose any special problem.

It can be shown quite readily that the power factor of the power supplied by an asynchronous generator is fixed by the amount of power being supplied and by the constants of the machine. One can reason this out by considering that an induction motor is started in the usual way and is brought up to its normal running speed which would be slightly below synchronous speed. The electrical input from the supply will consist of the magnetizing current, which is in quadrature with the voltage, and also the power component of current which is required to supply the rotational losses associated with the induction machine and whatever is on the same shaft. In order to have asynchronous operation we would have to assume that what is on the shaft is a turbine or drive of some sort. If we now introduce energy into the turbine the turbine speed will increase and some of the induction machine and shaft losses will be picked up by the driving turbine and the power component of current for the induction machine from the source will decrease. At some point, i.e., when we go through zero slip, the power component from the supply would fall to zero. However, we would still take the quadrature component of current which is necessary to supply the excitation requirements of the induction machine. As the speed of the turbine is further increased the torque, and power, become negative - which is to say they are flowing from the asynchronous machine stator back into the line. However, the excitation requirements remain substantially constant. Thus, the power factor of power supplied by the asynchronous generator is determined by the load which it is supplying and the constants of the machine itself.

Induction machines operating as generators have a very limited field of usefulness in electrical power systems. The inability to control the flow of reactive power is one of the reasons why they do have limited usefulness. However, they do have some positive advantages which can be considered in an evaluation involving asynchronous machines. One of the advantages is the fact that an induction, or asynchronous generator, does not have to be synchronized prior to connecting it into the system as does a synchronous machine. The machine can be connected to the system and running at no load for an indefinite period of time. It is instantly available for supply and emergency overload because all that is necessary is to admit energy into the prime mover connected to the same shaft. Also, it is suitable for high speed operation because of the simplcity and ruggedness of the rotor construction and since its voltage and frquency are controlled by the electrical system with which it is paralleled, it requires no attention to the synchronizing problem. Therefore it is suited to automatic or isolated hydraulic plants controlled from remote locations. There is no hunting, or oscillation problem, because of the asynchronous property of the machine itself. If the machine is short circuited the voltage at the terminals of the machine will drop which will reduce the excitation and will limit the short circuit current. If a dead short circuit occurs at the terminals, there will be a transient short circuit current but the steady state value of the short current has to be zero because there is no excitation in the steady state short circuit condition. If an induction motor is used to power something like an electric locomotive, a crane hoist, etc. which has the capability of running at a speed greater than zero slip, the induction machine is available as a generator if the speed does go above synchronous speed and thus it can provide a regenerative braking action where required.

It should be noted that the governor required for a prime mover which is to drive an asynchronous generator must be different from the governor and control system for a synchronous generator drive. This is so because provision must be made for varying the speed of the asynchronous machine in order to make it assume the desired amount of real load.

VII.8 STARTING PROBLEMS: Reference to the equivalent circuit for the induction machine with slip equal to 1, i.e., the rotor stationary indicates that minimum impedance exists in the circuit to limit the current into the machine from the supply circuit. Under normal conditions, wherein the rotor is free to turn, torque developed will immediately accelerate the rotor and, as the rotor accelerates and the slip decreases, the impedance increases and current decreases continuously until the steady state running condition is reached. Thus the current in-rush is a transient condition and does not persist any longer than the time required for the motor to accelerate from standstill to near normal running speed. However, this in-rush current can be on the order of five to eight times normal running current and can create a problem for the electrical system supplying the motor. This high current can cause excessive voltage dips on the circuits supplying the motor, resulting in light flicker for lighting loads and it introduces the possibility of magnetic contactors connected to the system becoming deenergized and disconnecting other connected loads. It is necessary, in an application of a relatively large induction motor, to examine carefully the effect that the motor starting will have on the rest of the system.

The basic problem is to reduce the in-rush current if its magnitude will be objectionable. This can be done either by inserting additional impedance in the circuit, external to the stator or in the rotor, or to reduce the voltage applied to the motor during the start.

If the motor is a wound rotor type, insertion of external resistance in the rotor circuit is relatively easy and this serves not only to limit the current but also the amount of resistance can be selected to yield very high torques under the starting condition (refer to VII-98). If the rotor is a squirrel cage rotor the additional external impedance must be placed in the stator circuit. This serves to limit the current but it also results in reduced voltage across the motor stator and it should be recalled that the torque developed is proportional to the square of the applied voltage. Use of a transformer to provide reduced voltage on starting will reduce the in-rush current but it also reduces the available torque. It will be shown below that the torque reduction is much less than the reduction suffered with inserted impedance for the same reduction in current from the supply.

Consider a motor with locked rotor impedance, Z_m, of 0.20 p.u. When 1.0 p.u. voltage is applied, a starting current of 5.0 p.u. results. Suppose it is desirable to reduce this current to 2.0 p.u. Total circuit impedance, which is the sum of Z_m and Z_e, external impedance, must be:

$$Z_m + Z_e = \frac{V}{I} = \frac{1.0}{2.0} = 0.5 \text{ p.u.} \qquad \text{(VII-119)}$$

from which $Z_e = 0.5 - Z_m = 0.5 - 0.2 = 0.3$ p.u.

The voltage across the motor, i.e., the voltage across Z_m is:

$$V_m = I_m Z_m = (2.0)(0.2) = 0.4 \text{ p.u.} \qquad \text{(VII-120)}$$

Since $T \propto V^2$, the starting torque, under this condition, is:

$$T_s = (0.4)^2 \, T_{sfv} = 0.16 \, T_{sfv} \text{ p.u.,} \tag{VII-121}$$

where T_{sfv} = torque developed with "full", or rated,voltage starting.

Suppose an auto transformer was inserted between the supply, $V = 1.0$, and the motor. Let the turns ratio of the auto transformer be:

$$\frac{N_1}{N_2} \quad \text{where} \quad N_1 > N_2 \tag{VII-122}$$

The motor would have an apparent impedance, Z'_m, to the source of:

$$Z'_m = \left(\frac{N_1}{N_2}\right)^2 Z_m \tag{VII-123}$$

If the "apparent value" was 0.5 p.u. the starting current (from the source) would be 2.0 p.u. Thus from (VII-123):

$$\frac{N_1}{N_2} = \sqrt{\frac{0.5}{0.2}} = 1.58 \tag{VII-124}$$

The voltage across Z_m would be, from (VII-123):

$$V_2 = \frac{N_2}{N_1} \, V_1 = \frac{1.0}{1.58} = 0.632 \text{ p.u.} \tag{VII-125}$$

and the starting torque is:

$$T_s = (0.632)^2 \, T_{sfv} = 0.4 \, T_{sfv} \text{ p.u.} \tag{VII-126}$$

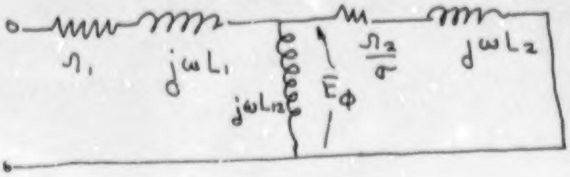
Compare this value with that obtained by external reactance starting, i.e., (VII-121) which results in the same value of starting current drawn from the supply! Of course, an autotransformer costs more than a simple impedance but it may be necessary to spend the extra money in order to have the additional starting torque on starting for a given supply in-rush current restriction.

VII.9 VARIABLE FREQUENCY OPERATION: Variable speed motors are widely used in industry. One reason for the widespread usage of the d.c. shunt motor is the ease with which it can be varied in speed. This usage is in spite of the relatively high maintanance associated with the brushes on the d.c. machine. Improvements in power semiconductor technology have made possible variable frequency supply from fixed frequency sources.

Since, in induction or synchronous machines, speed is determined from the applied frequency, the availability of variable frequency makes the induction motor a serious competitor to the d.c. shunt machine in variable speed applications. A variable frequency inverter is certainly more complex and expensive than an a.c. to d.c variable voltage converter but the ruggedness and cost of the squirrel cage induction motor are attractive enough to often offset the additional cost of the power supply.

Several simplifying assumptions will be made in the analysis of an asynchronous machine subject to variable frequency. These assumptions will be detailed as the derivation proceeds.

Figure (VII-8) portrays the equivalent circuit of an asynchronous machine with all variables in the rotor referred to the stator, and both stator and rotor quantities at stator frequency.



ω = stator angular velocity

Figure VII-8. Induction Motor Equivalent Circuit

The power transferred to the rotor is:

$$P_g = I_2^2 \frac{r_2}{\sigma} = \frac{E_\phi^2}{(\frac{r_2}{\sigma})^2 + (\omega L_2)^2} (\frac{r_2}{\sigma}) \qquad \text{(VII-127)}$$

$$= \frac{E_\phi^2}{\omega^2 L_2^2} (\frac{r_2}{\sigma}) \left(\frac{1}{\frac{r_2^2}{\sigma^2 \omega^2 L_2^2} + 1} \right) \qquad \text{(VII-128)}$$

Now, the rotor frequency, or electrical angular velocity, ω_r, is given by:

$$\omega_r = \sigma\omega \qquad \text{(VII-129)}$$

and we will define a time constant, τ :

$$\tau = \frac{L_2}{r_2} \qquad \text{(VII-130)}$$

(VII-128) becomes:

$$P_g = \frac{E_\phi^{\,2}(r_2)(\sigma\omega)^2 \quad \tau^2}{(\omega L_2)(L_2)(\sigma\omega)[1 + \tau^2(\sigma\omega)^2]} = \frac{E_\phi^{\,2}\,\tau\,\omega_r}{\omega L_2\left(1 + \tau^2\omega_r^2\right)} \qquad \text{(VII-131)}$$

If T = synchronous torque across the air gap to the rotor:

$$T = \frac{P_g}{\omega}\left(\frac{P}{2}\right) \qquad \text{(VII-132)}$$

Since mechanical angular velocity = $\frac{\omega}{P/2}$ where P/2 = number of pairs of poles.

With (VII-131), (VII-132) becomes:

$$T = \left(\frac{E_\phi}{\omega}\right)^2 \left(\frac{1}{L_2}\right)\left(\frac{\tau\omega_r}{1 + \tau^2\omega_r^{\,2}}\right)\frac{P}{2} \qquad \text{(VII-133)}$$

To normalize the torque, we will determine, ω_r, for maximum torque. Thus:

$$\frac{dT}{d\omega_r} = \left(\frac{P}{2}\right)\left(\frac{E_\phi}{\omega}\right)^2 \left(\frac{1}{L_2}\right)' \left(\frac{(1 + \tau^2\omega_r^{\,2})\,\tau - \tau\omega_r(2\tau^2\ \omega_r)}{(1 + \tau^2\omega_r^{\,2})^2}\right) = 0 \qquad \text{(VII-134)}$$

Solving for ω_r indicates that maximum torque occurs when:

$$\omega_r = \frac{1}{\tau} \qquad \text{(VII-135)}$$

If the value of ω_r from (VII-135) is substituted in (VII-133), maximum torque, T_m, is determined as:

$$T_m = \frac{1}{2}\ \frac{E_\phi^{\,2}}{\omega^2\ L_2}\left(\frac{P}{2}\right) \qquad \text{(VII-136)}$$

Substiuting, from (VII-136):

$$\frac{E_\phi^{\,2}}{\omega^2 L_2}\left(\frac{P}{2}\right) = 2\ T_m \qquad \text{(VII-137)}$$

into (VII-133) yields:

$$\frac{T}{T_m} = \frac{2\tau\omega_r}{1+\tau^2\omega_r^2} \qquad \text{(VII-138)}$$

We see that maximum torque is independent of r_2 (as shown previously) and occurs at

$$\omega_r = \frac{1}{\tau} = \frac{r_2}{L_2} \qquad \text{(VII-139)}$$

Since $\omega_r = \sigma\omega$, we find that maximum torque occurs at a slip, σ_m, of:

$$\sigma_m = \frac{r_2}{\omega L_2} \qquad \text{(VII-140)}$$

Contrast this with the value of σ_m from (VII-98), i.e.,

$$\sigma_m = \frac{r_2}{\sqrt{r_1^2 + (L_1+L_2)^2\,\omega^2}} \qquad \text{(VII-98)}$$

The difference between (VII-98) and (VII-140) is that r_1, L_1 are 'neglected'. This is another way of saying that $E_\phi = V_a$. In the expression above, $\tau\omega_r$ is normalized slip. (VII-138) in normalized form is plotted in Figure VII-9.

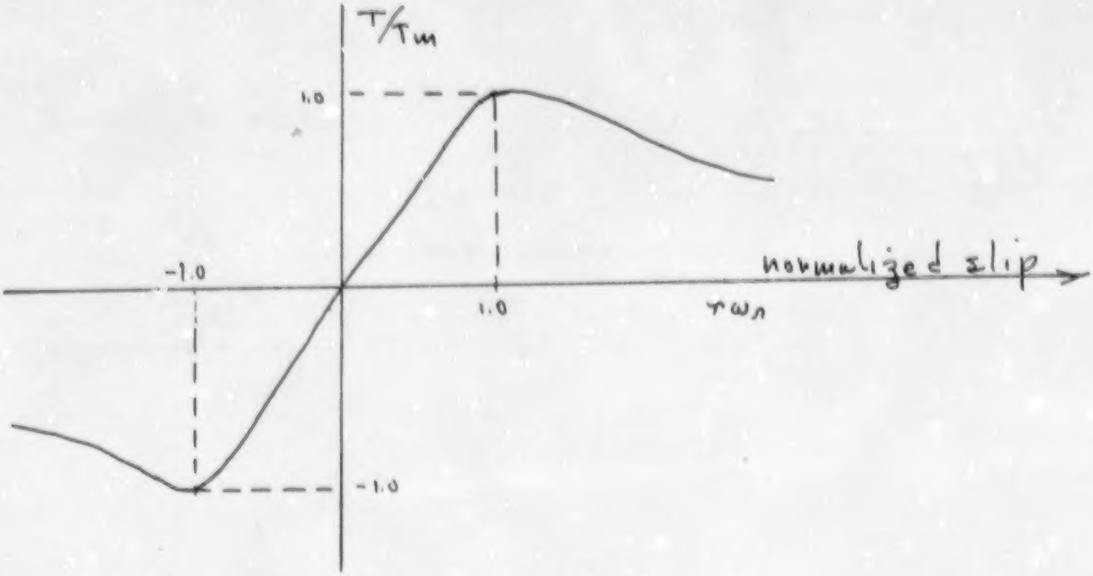


Figure VII-9. Normalized Torque Frequency Characteristic

A fundamental relationship of an asynchronous machine is that:

$$\omega_r = \omega - \omega_m \qquad \text{(VII-141)}$$

where ω_m = rotor speed in elec circuit rad/sec.

From (VII-139) and (VII-141) maximum torque is produced when the machine speed is:

$$\omega_m = \omega - \frac{1}{\tau} \qquad \text{(VII-142)}$$

Also, of course, this equation indicates that the motor can be adjusted to yield maximum torque at any desired speed ω_m including zero, if the drive frequency or stator frequency is set at:

$$f = \frac{\omega_m + \tau^{-1}}{2\pi} \qquad \text{(VII-143)}$$

The purpose of the derivations involving variable frequency operation is to permit us to visualize a drive system wherein either torque, T, or motor speed, ω_m can be varied (as desired) between zero and maximum. It will be assumed that the drive system includes a three phase source at constant frequency powering a squirrel cage motor with a small tachometer generator mounted on the shaft. The tachometer generator in its simplest form would be a permanent magnet excited generator with the same number of poles asthe induction motor. Theoutput voltage and frequency are proportional to the induction motor speed. The output is rectified to yield a dc voltage proportional to speed, ω_m.

A solid state frequency converter accepts the fixed frequency power from the three phase source and converts in into power at a specific frequency and voltage as determined by the control circuitry.

The power output from the frequency converter is impressed upon the induction motor. Note that both frequency and voltage must be controlled because, for an a.c. device the volts/cycle must be held constant for constant flux density (refer to equation (II-30).

Two possible inputs are desired speed or desired torque.

We will examine a system based on maintaining constant torque. From (VII-138) we can solve for a value of rotor angular velocity corresponding to a desired torque. Thus

$$\omega_r = \frac{1}{\tau}\left(\frac{T_m}{T} \pm \sqrt{\left(\frac{T_m}{T}\right)^2 - 1}\right) \qquad \text{(VII-144)}$$

An analog circuit for obtaining ω_r is as shown in Figure VII-10.

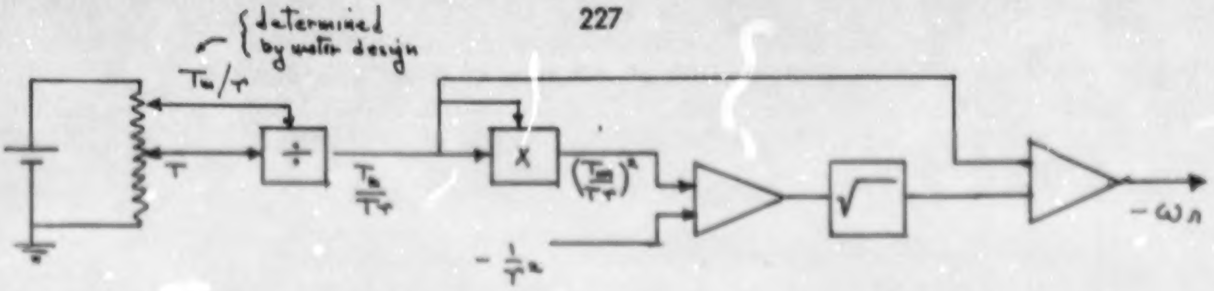

determined by motor design
Tm/τ
T
÷
Tm/Tτ
X
(Tm/Tτ)²
−1/τ²
√
−ωr

Figure VII-10. Analog Circuit to Obtain ω_r from T Desired

The $\pm$ sign in (VII-144) corresponds to values of $\tau\omega_r \leq 1.0$ or $\omega_r \geq 1.0$. Normal operating range for induction motor is for $\tau\omega_r \leq 1.0$, therefore the minus sign is chosen. $+\omega_r$ is added to the actual signal ω_m proportional to motor speed. This yields, from (VII-141), a signal corresponding to desired stator frequency. This analog signal is converted to digital form and compared with the digital value of actual converter frequency. The difference is used to increase or decrease the gate pulses to the converter. Similarly, the actual volts/cycle output is compared to a reference value based on the machine design, digitized and used to raise or lower the output voltage of the converter.

A simplified block diagram of a converter is shown in Figure VII-11.

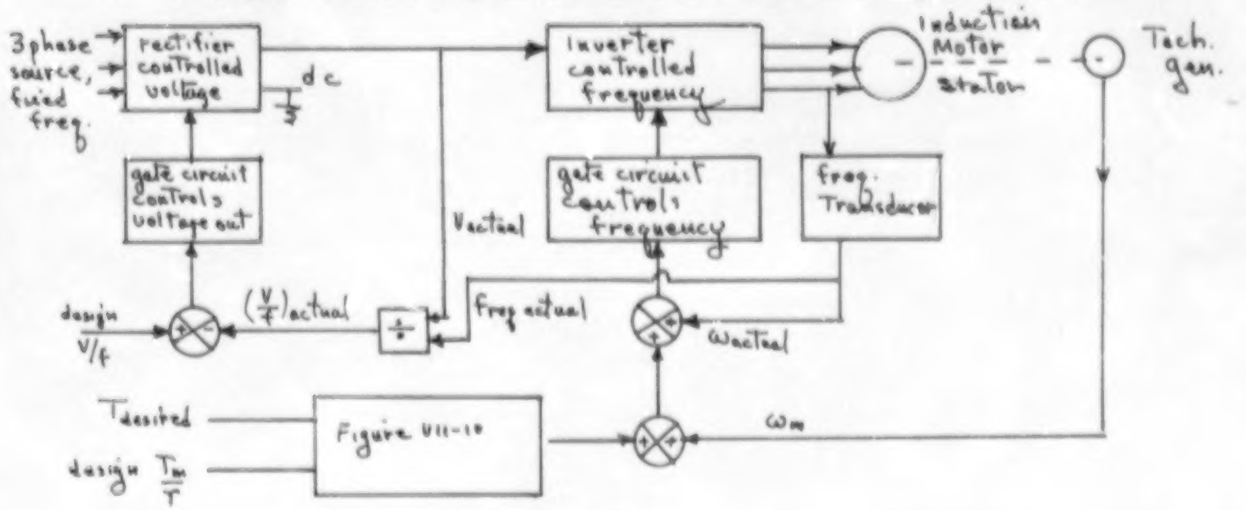

3 phase source, fixed freq.
rectifier controlled voltage
d c
inverter controlled frequency
Induction Motor stator
Tach. gen.
gate circuit controls voltage out
Vactual
gate circuit controls frequency
freq. Transducer
design V/f
(V/f)actual
÷
freq actual
ωactual
Tdesired
design Tm/T
Figure VII-10
ωm

Figure VII-11. Variable Speed Drive, Constant Torque Control

The speed torque characteristic of this type of drive is as shown in Figure VII-12.

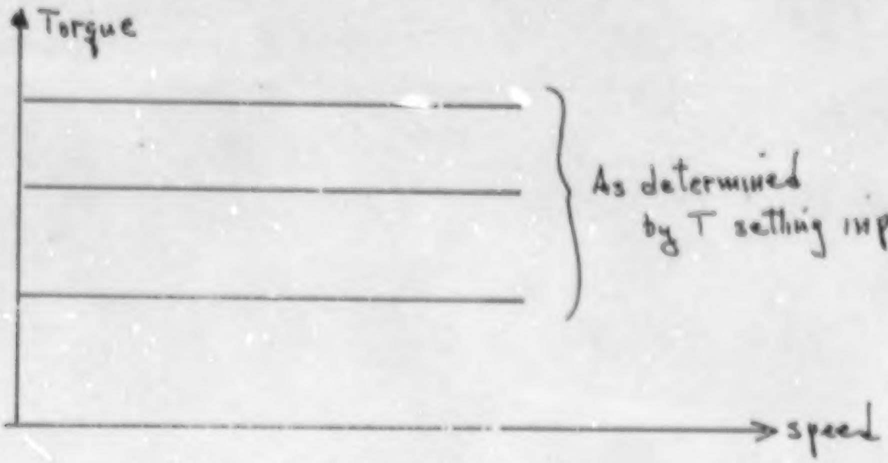


Figure VII-12. Speed-Torque Characteristic

Details of the rectifier-inverter and the gating circuits are not developed in this text. Also, of course, various sophistications of circuitry and combining of device functions are possible. The system is shown only in outline form to indicate possibilities.

APPENDIX I

Selected Reference Textbooks

A. Generalized Approach to Electromechanical Energy Conversion

Chapman, "Electromechanical Energy Conversion", Blaisdell, 1965.

Crosno, "Fundamentals of Electromechanical Conversion", Harcourt Brace and World, 1968.

Gourishankar, "Electromechanical Energy Conversion", International Textbook Co, 1965.

Levi and Panzer, "Electromechanical Power Conversion", McGraw-Hill, 1966.

Majmudar, "Electromechanical Energy Converters", Allyn & Bacon, 1965.

Meisel, "Electromechanical Energy Conversion", McGraw-Hill, 1966.

Nasar, "Electromagnetic Energy Conversion Devices and Systems," Prentice Hall, 1970.

O'Kelly and Simmonds, "Introduction to Generalized Machine Theory", McGraw-Hill, 1968.

Seely, "Electromechanical Energy Conversion", McGraw-Hill, 1962.

Sleman, "Magnetoelectric Devices", Wiley 1966.

White and Woodson, "Electromagnetic Energy Conversion", Wiley, 1959.

B. Texts Emphasizing Electric Machines

Adkins, "General Theory of Electric Machines", Wiley 1957.

Alger, "The Nature of Induction Machines", Gordon and Breach, 1965.

Botvinnik, "Asynchronous Synchronous Machines", Macmillan, 1964.

Carr, "Electric Machinery", Wiley 1958.

Concordia, "Synchronous Machines", Wiley.

Fitzgerald and Kingsley, "Electric Machinery", McGraw-Hill, 2nd edition.

Gibbs, "Tensors in Electric Machine Theory", Chapman and Hall, 1952.

Hague, "The Principles of Electromagnetian Applied to Electric Machines", Dover, 1962.

Jevons, "Electrical Machine Theory", American Elsevier, 1966.

Jones, "The Unified Theory of Electric Machines", Plenum Press, 1967.

Kloeffler, et. al., "Direct Current Machinery", Macmillan 1934.

Kron, "Equivalent Circuits of Electric Machinery", Dover 1967.

Ku, "Electric Energy Conversion", Ronald 1959.

Kuhlman, "Design of Electrical Apparatus", 3rd Edition, Wiley, 1950.

Langsdorf, Theory of Alternating Current Machinery, McGraw-Hill, 1955, 2nd Edition.

Lawrence and Richards, "Principles of Alternating Current Machinery", McGraw-Hill, 1953, 4th edition.

Liwschitz-Garik and Whipple, "Direct Current Machines", Van Nostrand, 1956, 2nd edition.

Liwschitz-Garik and Whipple, "Alternating Current Machines", Van Nostrand, 1961, 2nd edition.

Lyon, "Transient Analysis of Alternating Current Machinery", Wiley 1954.

Puchstein, Lloyd, Conrad, "Alternating Current Machines", Wiley 1954, 3rd edition.

Pestarini, "Metadyne Statics", MIT Press, 1952.

Sah, "Fundamentals of Alternating Current Machines", McGraw-Hill, 1936.

Thaler and Wilcox, "Electric Machines", Wiley 1966.

Lewis, "Principles of Synchronous Machinery", ITT Press, 1959.

APPENDIX II

Units Conversion Multipliers and Constants

- Length -

1 meter = 3.281 feet = 39.37 inches

-- Mass -

1 kilogram = 2.205 lb.

- Force -

1 newton = 0.225 lb.

- Torque -

1 newton-meter = 0.738 ft.-lbs.

- Energy -

1 joule = 1 watt-sec = 0.738 ft.lbs.

- Power -

1 watt = 1.341 x 10^{-3} horsepower

- Moment of Inertia -

1 kilogram-meter^2 = 23.7 lb.-ft.^{2}

- Magnet Relationship -

1 ampere turn/meter = 0.0254 ampere turn/inch

1 weber/meter^2 = 10000 gauss = 64.5 kilolines/inch^2

1 weber = 10^8 maxwells = 10^8 lines

Permeability of Free Space = μ_o = $4\pi \times 10^7$ weber/ampere turn meter
Permittivity of Free Space = ε_o = 8.854×10^{-12} coulomb^2/newton meter^2
Gravity effect = g = 32.2 ft./sec.^{2} = 9.807 meter/sec^2